A Dynamic Interference Model for Benham's Top

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The study of subjective color in perceptual psychology began with Prévost in 1826, followed by Fechner in 1839 and Benham in 1894 [1]. Charles Benham (1860–1929) invented Benham's top, in which colors emerge from a black-and-white design. This toy became very popular in England and was featured in the scientific journal *Nature*, but the cause of the subjective color remains unknown [2].

From Polar to Cartesian Coordinates

Benham's Top is a disk with a diameter of around 10 cm, one half painted solid black and the other having arcs of three lines each (**Fig. 1**). There are four blocks of arcs, each with a 45° central angle.

Fig. 1: Benham's top.

Giving the top a fast spin, at first you see nothing, but as its rotational speed slows the pattern begins to flicker, and colors immediately appear. When spinning the top clockwise (to the right), colors appear in the order blue, green, orange, and red from the periphery inward (**Fig. 2**-1). Spinning the top in the counterclockwise direction results in the color order red, orange, green, blue (**Fig. 2**-2).

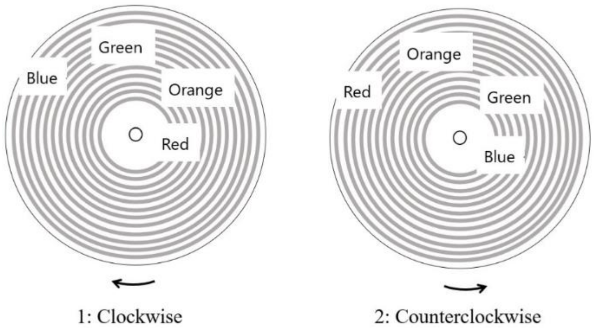

Fig. 2: Rotational directions and orders of colors.

I focused on the fact that green and orange appear in the middle of the four arc blocks, not on the periphery or in the center. Why do those colors flip when the top spins in opposite directions? In the Benham's top shown in **Fig. 3**-1, setting aside the black lower half, the arrangements of the green and orange arcs are radially reversed. The central angle is 360° in one rotation and the black in the lower half covers 180°. When the 180° in the upper half is divided into white, black, and white, we have clockwise

Green arc: Black (180°) — White (45°) — Black (45°) — White (90°)
Orange arc: Black (180°) — White (90°) — Black (45°) — White (45°)

For future development of this theory, I shifted the positions of the outer and central arcs in the standard Benham's top by about 20° around the central angle, as shown in **Fig. 3**-2, so that they do not touch the black area of the lower semicircle at the locations indicated by the arrows. In this way, all four blocks will have a black–white–black–white pattern, and the subjective colors, which previously had a grayish cast, become easier to see.

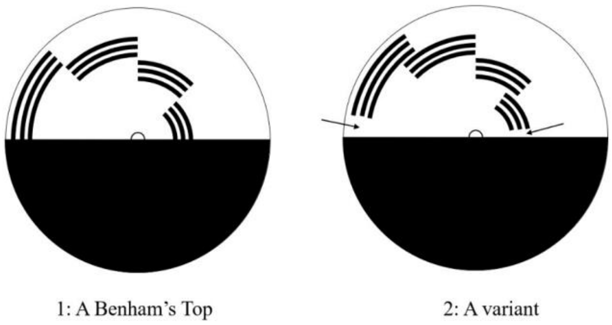

Fig. 3: A variant of Benham's top.

We will continue our discussion using the pattern in **Fig. 3**-2. Converting polar to Cartesian coordinates, we get the pattern shown in **Fig. 4**. Following the top's periphery counterclockwise from A to B, we can label the arcs as 1 through 4. We assume the disk is turning to the right (clockwise).

From the center of the disk, the subjective colors are red for arc 1, orange for arc 2, green for arc 3, and blue for arc 4. In Cartesian coordinates (the right side of **Fig. 4**), the colors are red, orange, green, and blue from top to bottom. Each of arcs 1 through 4 has the order black–white–black–white. In the Cartesian coordinates, the lengths of the first and second black parts are the same, but the lengths of the white parts before and after the second black part are different. If the first white provides a primary stimulus and the second a secondary stimulus, the two may produce some form of interference (reinforcement, cancellation, etc.).

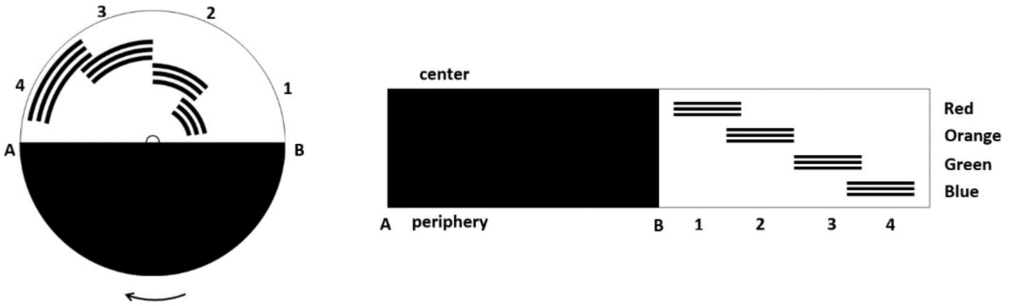

Fig. 4: Conversion from polar to Cartesian coordinates.

Dynamic Interference

The interference seen in Young's interference experiment and in the thin film of a soap bubble was due to differences in the paths via which light arrives. In the case of Benham's top, however, visual stimuli travel to the visual system from the eyes, then through the retina and on to the brain, where they are perceived as subjective color, with nothing along that route corresponding to a path difference. Everything follows the same route, so interference seemingly cannot occur. I will therefore show that here, interference occurs as a "phase difference" rather than "path difference."

In 1979, I thought that colors might emerge when a primary stimulus was delayed for some reason, causing It to overlap with a secondary stimulus (**Fig. 5**). [3] I speculated that the magnitude of the primary stimulus delay might be related to the wavelength of the emerging color, and inversely proportional to the strength of the primary stimulus, but I did not arrive at a theory.

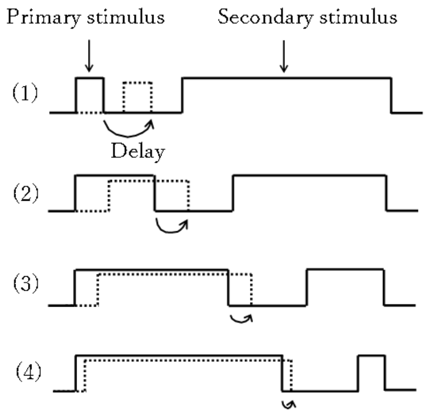

Fig. 5: Interference between primary and secondary stimuli.

Let us show that a phase shift can cause two waves to reinforce or cancel each other. Consider two sinusoidal waves having the same wavelength and amplitude and at phase difference α:

$$y_1 = \sin(\theta + \alpha)$$
$$y_2 = \sin\theta$$

Fig. 6 shows an overview of the overlap. The sine waves for the primary and secondary stimuli in the top row of that figure have the same amplitude and wavelength, and there are five cycles. When there is no phase difference, superposing the two waves produces a sine wave with the same wavelength but twice the amplitude. When superposing the primary wave onto the secondary wave at a phase shift of 90°, the maximum amplitude of the wave becomes 1.4 times larger, when shifted 180°, the waves cancel and the amplitude becomes zero, and a 360° shift again doubles the amplitude.

By the formula for the sum-product of trigonometric functions, the superposition of two sine waves is

$$y_1 + y_2 = \sin(\theta + \alpha) + \sin\theta = 2\cos\frac{\alpha}{2}\sin\left(\theta + \frac{\alpha}{2}\right).$$

White contains all colors, but to simplify the discussion, we consider only red, green, and blue. **Fig. 7** shows twelve graphs in three rows and four columns. The leftmost column shows waves with four red cycles, five green cycles, and six blue cycles. Superposing all three would produce white. Each row shows superpositions of the waves providing primary and secondary stimuli with the phase shifted 360° at the red wavelength, 360° at the green wavelength, and 360° at the blue wavelength.

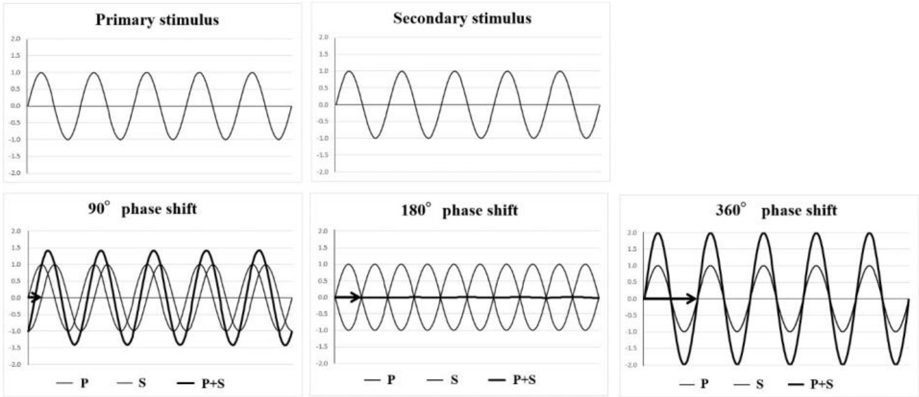

Fig. 6: Wave superpositions. (P: primary stimulus; S: secondary stimulus).

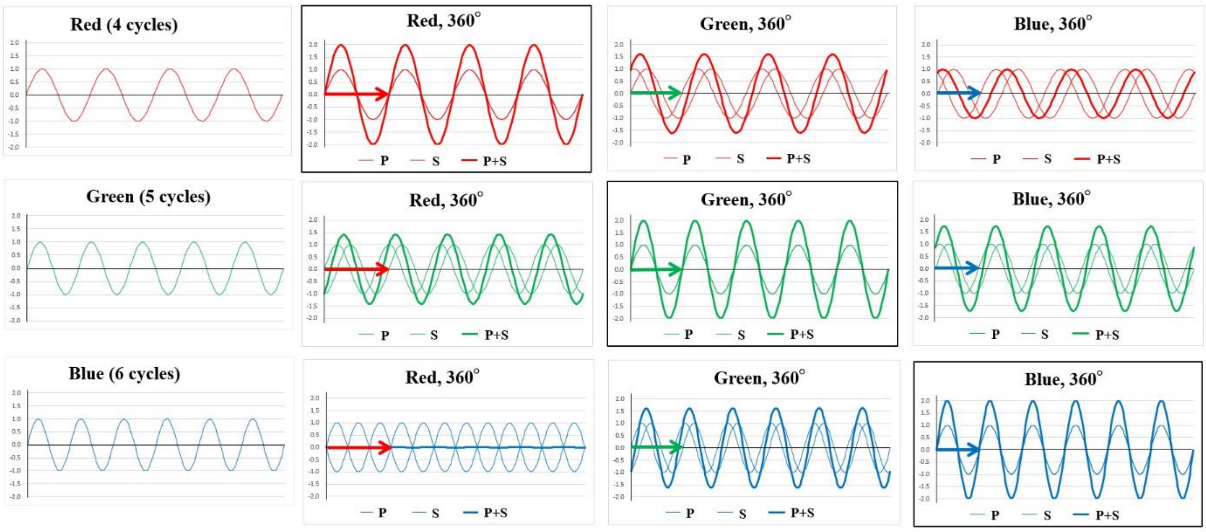

Fig. 7: Why certain colors emerge (P: primary stimulus; S: secondary stimulus).

Let's look at **Fig. 7** in the columnar (vertical) direction. The first column from the left superimposes red, green, and blue, producing white. In the second column, shifting and superimposing the red wavelength by 360° doubles the red amplitude, increases the green amplitude by a factor of 1.4, and makes the blue amplitude become 0, making the red color predominantly emerge. In the third column, shifting and superimposing the green wavelength by 360° increases the red amplitude by 1.6, doubles the green amplitude, and increases the blue amplitude by 1.6, making the green color predominantly emerge. In the fourth column, shifting and superimposing the blue wavelength by 360° leaves the red amplitude the same, increases the green amplitude by 1.7, and doubles the blue amplitude, making the blue color predominantly emerge.

That result can be used to estimate subjective colors. Taking the three primary colors as RGB, the magnitude of the amplitude due to superposition in the columnar (vertical) direction in **Fig. 7** is as follows:

(R, G, B) = (2, 1.4, 0),

(R, G, B) = (1.6, 2, 1.6),

(R, G, B) = (1, 1.7, 2).

Recalculating these as chromaticity values of 0 to 255, we get

(R, G, B) = (255, 179, 0),

(R, G, B) = (204, 255, 204),

(R, G, B) = (128, 217, 255).

Fig. 8 shows a plot based on these figures.

The top row is (R, G, B) = (255, 255, 255), which is white due to additive mixing of the three primary colors. The second row shows the subjective color with only the red wavelength phase-shifted, the third row shows the subjective color with only the green wavelength phase-shifted, and the fourth row shows the subjective color with only the blue wavelength phase-shifted. Subjective colors are intermediate colors, more like pastels than vivid reds, greens, and blues.

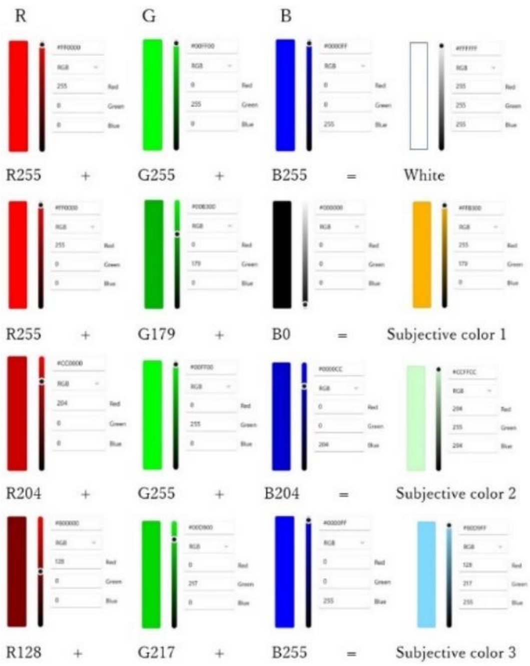

Fig. 8: Subjective color estimation with a dynamic interference model.

Equations of motion for delay

Fig. 7 shows why certain colors emerge, but let us consider how this corresponds to a Benham's top. In **Fig. 4** the rotating disk is converted to Cartesian coordinates. This causes the three arcs to become three straight lines, but I used three lines only to make the subjective colors appear more readily. The same effect would occur with a single line, which can be simplified as shown in **Fig. 9**.

As the top rotates, in Cartesian coordinates it moves from left to right, then repeats. I call the first white segment the primary stimulus and the second white segment, between the black ones, the secondary stimulus. Dynamic interference occurs when the primary stimulus shifts and overlaps the secondary stimulus. If the magnitude of the primary stimulus's shift is largest for Red (1) and smallest for Blue (4), that corresponds to the reason why a particular color emerges in **Fig. 7**. But does such a convenient hypothesis really hold up?

Therefore, I made a bold hypothesis. Consider white primary stimuli as the amount of light, the substance being transmitted, and the object in motion, with masses $m\ (m_1, m_2, m_3, m_4)$ and accelerations $a\ (a_1, a_2, a_3, a_4)$.

If the white area is large, the amount of light is large and the mass is large; if that area is small, the amount of light and mass are small. Black areas reflect no white light and convey nothing to the visual system. Rather, they may function to stop or delay the progression of physical quantities conveyed to the visual system by preceding white areas. Since the length of each black segment is the same, we can let the force be F (a constant), and we obtain for each the equation of motion:

$$F = ma = m_i a_i.$$

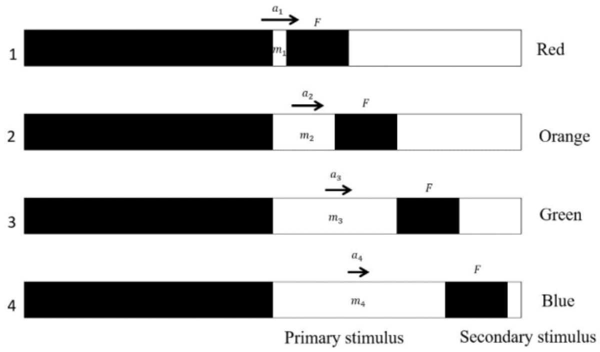

Fig. 9: Equation of motion for delay ($F = ma$).

From **Fig. 9**, the mass m has the relation

$$m_1 < m_2 < m_3 < m_4.$$

Since the F in the equation of motion $F = ma$ is constant, the acceleration a has the relation

$$a_1 > a_2 > a_3 > a_4.$$

Newton's equations of motion state that the lighter an object, the easier it is to move, and the heavier it is, the harder it is to move, and this also holds true for Benham's top. This can be described for red (1) as follows: The mass m_1 of the primary stimulus is the smallest, so the acceleration a_1 is the largest, and thus the primary stimulus experiences the largest shift and overlaps and interferes with the secondary stimulus. As the second column in **Fig. 7** shows, the longest wavelength, red, emerges.

Conclusion

It has been nearly two centuries since Prévost discovered subjective color in 1826. A number of black-and-white patterns have been devised from which subjective color appears, but the cause of this phenomenon remains unknown. I think this is ultimately a consequence of the model of subjective color expression shown in **Fig. 10**. Letting a, b, c, d be the respective lengths of the black–white–black–white segments, when this pattern is repeated *n* times per second, it seems that varying the values of these five variables produces a variety of colors. To test this, I would like to have not a spinning disk like a Benham's top, but an experimental apparatus that can display black–white–black–white patterns on a screen. Proving my hypothesis would be a major advancement in the study of subjective color, following Prévost, Fechner, and Benham [1, 2].

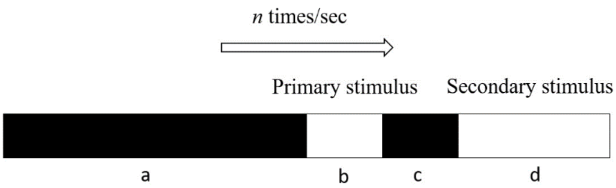

Fig. 10: A model of subjective color expression.

References

1) J. Cohen and D. A. Gordon: The Prevost–Fechner–Benham subjective colors, *Psychological Bulletin*, **46**(2), 97–136, 1949.

2) C. Campenhausen and J. Schramme: 100 Years of Benham's Top in Colour Science, *Perception*, **24**(6), 695–717, 1995.

3) Y. Nishiyama: The Mathematics of Benham's Top, *International Journal of Pure and Applied Mathematics.*, **93**(3), 399–408, 2014.