

Machin's Formula and Pi

Yutaka Nishiyama

Abstract

This article explains the calculation of Pi historically, focusing on Machin's formula. Archimedes' formula is shown first, followed by Machin's formula using Gregory's formula. Machin's formula makes particularly ingenious use of the tan double and quadruple angle trigonometric addition formulae. The chapter closes with an explanation of Takano's formula.

Keywords : Pi, Machin's formula, Archimedes' method, Gregory's formula, Pythagorean triangles, Takano's formula

1. Finding Pi to 1000 decimal places

I have been working on and researching computer-related topics since 1970 – for almost 40 years. I never really had much interest in the calculating Pi, the so-called circle ratio. The competition between Japan and America to compute Pi to more and more digits was ongoing, but it was an issue of relevance only to people with access to supercomputers, and I thought it held no interest for the average person. Recently, Prof. Yoshio Kimura sent me an enjoyable book called 'Playing with Simple Computer Programming' (see Kimura, 2003). Chapter 8 contains an explanation of the calculation of Pi using Decimal BASIC, according to which, with 50 or so lines of code, Pi could be obtained to 100 decimal places. Recent personal computers have better performance, and computing Pi to 1000 decimal places is now an easy matter.

The equation used here is known as Machin's formula.

$$\frac{\pi}{4} = 4 \arctan \frac{1}{5} - \arctan \frac{1}{239}$$

This equation was discovered in 1706 by John Machin, and has been in use for around 300 years. It is suitable for the calculation of Pi because it converges quickly. In this chapter, let's think about how this equation was introduced, and how it can be programmed into a computer.

2. Archimedes' method

Pi is the ratio of the length of the diameter of a circle to its circumference. At elementary school, we learn its value as 3.14. In practical terms 3 digits are sufficient, but the development of computers and advances in mathematics have worked together to update the value of Pi to a remarkable number of places. First of all, let's take a look at the methods that were adopted before the advent of the calculator. Archimedes, from ancient Greece, calculated Pi using regular polygons which contact and either enclose, or are enclosed by a given circle. Suppose

the radius of a circle is 1 ($OA=OB=1$). For regular hexagons, the total lengths of the sides of the enclosed hexagon, and the enclosing hexagon are 6 and $4\sqrt{3}$ respectively. Thus, $6 < 2\pi < 4\sqrt{3}$ so $3 < \pi < 2\sqrt{3} \approx 3.464$ (Figure 1).

For regular dodecagons (with 12 sides), $OA=OB=OC=1$ and we may define $AB=a$ and $AC=b$. Denoting the midpoint of AB by M , setting $MC=x$, and applying Pythagoras' theorem to the triangles OAM and ACM , we have

$$1^2 = \left(\frac{a}{2}\right)^2 + (1-x)^2, \quad b^2 = \left(\frac{a}{2}\right)^2 + x^2.$$

Solving this confirms that the circle ratio $\pi > 6b \approx 3.105$ (Figure 2).

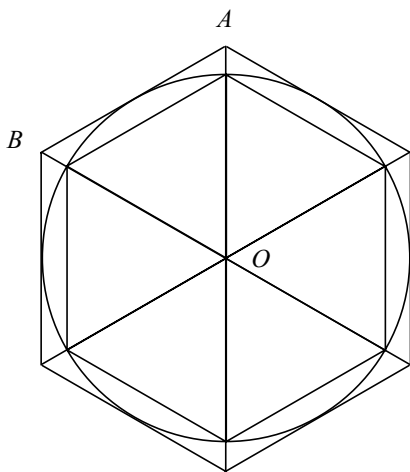


Figure 1. Regular hexagon

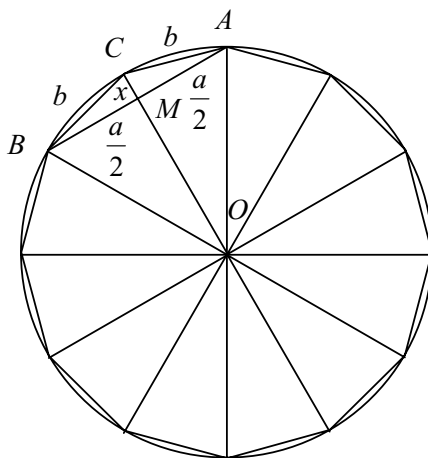


Figure 2. Regular dodecagon

It is also possible to compute the total side lengths for regular 24-sided and regular 96-sided polygons using Pythagoras' theorem. I'd like for those readers who are interested to confirm this for themselves. In the time of the ancient Greeks, who were not aware of irrational numbers, Archimedes used a 96-sided regular polygon, and calculated the value 3.14.

3. Gregory's formula

The period in which the value was evaluated using regular polygons continued until the 17th century. The number of digits was not increased using brute force calculation methods, as these merely increased the number of edges. This reform had to wait until the development of differential Calculus. In 1671, Gregory discovered what is known as Gregory's sequence.

$$\arctan(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1} = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \frac{1}{9}x^9 - \dots$$

In 1674, Leibnitz also made the same discovery independently, and the sequence is also called the Gregory-Leibnitz sequence. Substituting $x=1$ in this formula yields the following sequence.

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \dots$$

The convergence of this sequence is extremely slow, but by taking the partial sum, an approximation to the value of π may be obtained.

Let's take a look at how Gregory's sequence was introduced. Since Gregory's sequence is a Taylor expansion of $\arctan(x)$, let's take a look at this as well. For an infinitely differentiable function $f(x)$, the so-called Taylor series is the power series, $\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n$, and when this series has the same value as the original function, $f(x)$ is said to have a Taylor expansion. This may be thought of in terms of a neighborhood around $x=a$, and is known as the Taylor expansion around $x=a$. When $a=0$, the expansion is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

which particular case is also called the Maclaurin expansion.

$$f(x) = f(0) + f^{(1)}(0)x + \frac{f^{(2)}(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \dots$$

Leaving the rigorous proof of this equation for another time, isn't it possible to see intuitively that this formula is correct, when both sides are differentiated? \tan (tangent) and \arctan (arctangent) are mutually inverse functions and may be written as follows.

$$y = \tan x \left(-\frac{\pi}{2} < x < \frac{\pi}{2} \right)$$

$$x = \tan^{-1}y$$

The notation $y = \arctan x$ is also used. Putting $y = f(x) = \arctan x$, it is clear that,

$$f(0) = 0.$$

Let's think about the derivative of \arctan in order to calculate $f^{(1)}(0)$. Differentiating it in this form is difficult, so let's first consider

$$x = \tan y.$$

The derivative of x on the left-hand side is 1, and the right-hand side is a composite function so,

$$1 = (1 + \tan^2 y) \frac{dy}{dx}.$$

Solving this yields the 1st derivative.

$$\frac{dy}{dx} = \frac{1}{1 + \tan^2 y} = \frac{1}{1 + x^2}$$

This can be written

$$f^{(1)}(x) = \frac{1}{1 + x^2}.$$

The 2nd and 3rd derivatives calculated from this equation are as follows.

$$f^{(2)}(x) = \frac{-2x}{(1 + x^2)^2}$$

$$f^{(3)}(x) = \frac{2(3x^2 - 1)}{(1 + x^2)^3}$$

Substituting $x=0$ into these equations,

$$f^{(1)}(0)=1, f^{(2)}(0)=0, f^{(3)}(0)=-2.$$

Substituting these into the terms of the Taylor expansion yields the following equation.

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots$$

4. Machin's formula

The formula for the circle ratio using the arctangent discovered by the Englishman John Machin in 1706 was mentioned above. Let's restate Machin's formula.

$$\frac{\pi}{4} = 4 \arctan \frac{1}{5} - \arctan \frac{1}{239}$$

Machin used Gregory's series with this formula and obtained Pi to 100 decimal places. Allow me to explain the relationship between Machin's formula and Gregory's series.

By substituting $x = \frac{1}{5}$ or alternatively $x = \frac{1}{239}$ into Gregory's series,

$$\arctan \frac{1}{5} = \frac{1}{5} - \frac{1}{3 \cdot 5^3} + \frac{1}{5 \cdot 5^5} - \frac{1}{7 \cdot 5^7} + \dots$$

$$\arctan \frac{1}{239} = \frac{1}{239} - \frac{1}{3 \cdot 239^3} + \frac{1}{5 \cdot 239^5} - \dots$$

can be obtained. Substituting these into Machin's formula yields the following.

$$\begin{aligned} \frac{\pi}{4} &= 4 \arctan \frac{1}{5} - \arctan \frac{1}{239} \\ &= 4 \times \left(\frac{1}{5} - \frac{1}{3 \cdot 5^3} + \frac{1}{5 \cdot 5^5} - \frac{1}{7 \cdot 5^7} + \dots \right) \\ &\quad - \left(\frac{1}{239} - \frac{1}{3 \cdot 239^3} + \frac{1}{5 \cdot 239^5} - \frac{1}{7 \cdot 239^7} + \dots \right) \end{aligned}$$

This is how the circle ratio can be computed.

Now let's look at the details in Machin's formula. This formation of the equation can be explained with the elegant use of trigonometric addition formulae. The addition formula for \tan is an old friend that appears in high-school mathematics textbooks.

$$\tan(\alpha + \beta) = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta}$$

Setting $\alpha = \beta$ in this formula can be used to derive the double angle formula.

$$\tan 2\alpha = \frac{2 \tan \alpha}{1 - \tan^2 \alpha}$$

Setting $\tan \alpha = \frac{1}{5}$ means that $\alpha = \arctan \frac{1}{5}$, and the double angle and quadruple angle tangents can be obtained from the formulae above as follows.

$$\tan 2\alpha = \frac{2 \times \frac{1}{5}}{1 - \frac{1}{5} \times \frac{1}{5}} = \frac{10}{24} = \frac{5}{12}$$

$$\tan 4\alpha = \frac{2 \times \frac{5}{12}}{1 - \frac{5}{12} \times \frac{5}{12}} = \frac{2 \times 5 \times 12}{12 \times 12 - 5 \times 5} = \frac{120}{119}$$

$\tan 4\alpha$ is $\frac{120}{119}$, which is very close to 1, since it only differs by $\frac{1}{119}$. Calculating the tan of the difference between 4α and $\frac{\pi}{4}$ reveals the following attractive form

$$\begin{aligned} \tan\left(4\alpha - \frac{\pi}{4}\right) &= \frac{\tan 4\alpha - \tan \frac{\pi}{4}}{1 + \tan 4\alpha \tan \frac{\pi}{4}} = \frac{\frac{120}{119} - 1}{1 + \frac{120}{119} \times 1} \\ &= \frac{120 - 119}{119 + 120} = \frac{1}{239}. \end{aligned}$$

From this formula,

$$4\alpha - \frac{\pi}{4} = \arctan \frac{1}{239}, \text{ and}$$

$$\frac{\pi}{4} = 4 \arctan \frac{1}{5} - \arctan \frac{1}{239},$$

thus constituting a derivation of Machin's formula.

Let's think about Machin's formula from another angle. Beginning from $\tan \alpha = \frac{1}{5}$ we thought about the fact that $\tan 2\alpha = \frac{5}{12}$ and $\tan 4\alpha = \frac{120}{119}$. There is a triangle which corresponds to these, and it is shown in Figure 3.

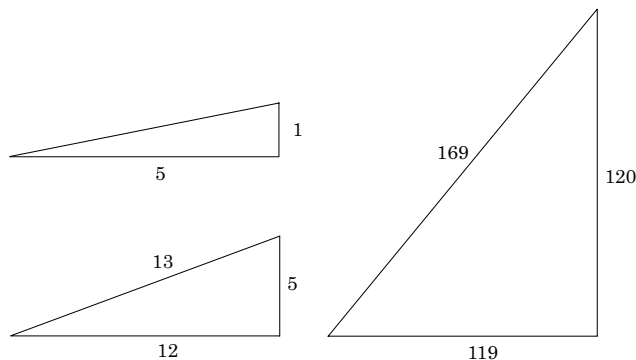


Figure 3. Double and quadruple angle triangles

In the double angle case, the edge lengths are in the ratio 12 : 5 : 13, and in the quadruple angle case they are 119 : 120 : 169. These values satisfy the relationship expressed by Pythagoras' theorem.

$$13^2 = 12^2 + 5^2, 169^2 = 119^2 + 120^2$$

The double and quadruple angle triangles are also called Pythagorean triangles. Pythagorean triangles satisfy the following relationship.

Theorem

For mutually prime natural numbers m, n , if $\tan \alpha = \frac{n}{m}$ ($m > n$), then the right-angled triangle which has an acute angle 2α is a Pythagorean triangle. Also, the ratio of the lengths of the three edges is $m^2 - n^2 : 2mn : m^2 + n^2$.

Substituting $m=5$ and $n=1$ into this theorem, the ratio of the lengths of the edges in the double angle triangle can be calculated from $m^2 - n^2 = 24$, $2mn = 10$ and $m^2 + n^2 = 26$, revealing the ratio $24 : 10 : 26 = 12 : 5 : 13$. Substituting $m=12$ and $n=5$ yields $m^2 - n^2 = 119$, $2mn = 120$ and $m^2 + n^2 = 169$ from which the quadruple angle triangle's edge length ratio can be calculated. Machin's formula makes really ingenious use of the tan double and quadruple angle trigonometric addition formulae. There may be many students who wonder whether the trigonometric and multiple angle formulae have much of a role in the mathematics that appears in examinations, but I'd like people to make a point of remembering the tremendous use made of these mathematical assets.

5. Around Machin's formula

Machin's formula was explained above, but regarding its derivation, just how the formula was discovered seems to be unknown. Perhaps Machin's formula was discovered by accident. Or perhaps it was obtained by building on a mathematical concept. Let's investigate whether there are any other formulae like Machin's. Consider the following series, where a_k represent integers which may be positive or negative, and b_k represent positive integers.

$$\frac{\pi}{4} = \sum_{k=1}^n a_k \arctan \frac{1}{b_k}$$

Regarding the values shown in the 2nd term, the following formulae are known.

- (1) $\frac{\pi}{4} = \arctan \frac{1}{2} + \arctan \frac{1}{3}$
- (2) $\frac{\pi}{4} = 2 \arctan \frac{1}{3} + \arctan \frac{1}{7}$
- (3) $\frac{\pi}{4} = 2 \arctan \frac{1}{2} - \arctan \frac{1}{7}$
- (4) $\frac{\pi}{4} = 4 \arctan \frac{1}{5} - \arctan \frac{1}{239}$

(1) was conceived by Euler, and can be proven using the tangent addition formula. Try this yourself. The type of problem shown in Figure 4 can sometimes be found in examination reference books. This problem requires one to prove that

$$\alpha = \beta + \gamma,$$

and the original version of this problem seems to be related to Euler's formula for the

calculation of Pi.

(4) is Machin's formula. When the values of a_k and b_k are small, it's possible to confirm using calculation by hand, but when the values are large, as in the case of Machin's formula, one must admit defeat. These days we have computers, so comprehensive search methods incorporating a computer program are possible, but it remains a question how Machin's formula was discovered in the 18th century when there were no computers.

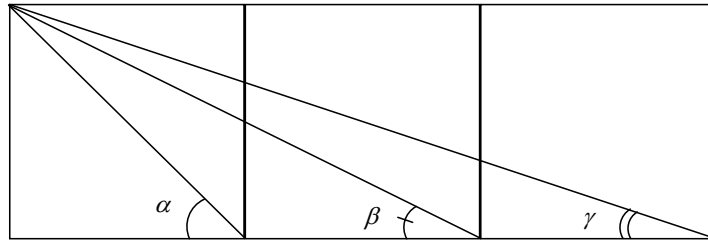


Figure 4. $\alpha = \beta + \gamma$

When the number of terms is increased to 3, the following forms may be expressed. (6) is Gauss' formula.

$$(5) \quad \frac{\pi}{4} = \arctan \frac{1}{2} + \arctan \frac{1}{5} + \arctan \frac{1}{8}$$

$$(6) \quad \frac{\pi}{4} = 12 \arctan \frac{1}{18} + 8 \arctan \frac{1}{57} - 5 \arctan \frac{1}{239}$$

6. Kikuo Takano's formula

Now, returning to Machin's formula, let's think about the actual calculation. Even despite knowing Machin's formula, there are some people who can only calculate Pi to 7 significant digits using a computer. Also, given a calculator which can only handle 8 digits, some people think it's only possible to calculate 8 digits. Using a calculator with 8 digits, in principle, it's possible to obtain Pi to 1000 digits, so let's look at the method.

Allow me to explain using $1 \div 239$. For the sake of brevity, let's calculate the digits 3 at a time.

$$\begin{aligned} 1 \div 239 &= 0 \text{ remainder } 1 \\ 1000 \div 239 &= 4 \text{ remainder } 44 \\ 44000 \div 239 &= 184 \text{ remainder } 24 \\ 24000 \div 239 &= 100 \text{ remainder } 100 \end{aligned}$$

By arranging these quotient parts we obtain the answer:

$$0. 004 \ 184 \ 100 \dots$$

Regarding the recent competition between Japan and America to calculate Pi, in 2002 Yasumasa Kaneda calculated 1 trillion, 2411 hundred million digits using Takano's formula (announced in 1982) on a HITACHI SR8000. See Takano (1983) for more information on his formula. With respect to Machin's formula, this formula is incomparably complicated. New formulae are discovered one after the other, but the use of computers is common. However,

computers alone are useless, and their foundations are based on many mathematical formulae.

$$\frac{\pi}{4} = 12 \arctan \frac{1}{49} + 32 \arctan \frac{1}{57} - 5 \arctan \frac{1}{239} + 12 \arctan \frac{1}{110443}$$

References

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