Fixed Points in Similarity Transformations

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Abstract

A new method of constructing fixed points in congruence transformations is introduced, and a detailed explanation of fixed points in similarity transformations then follows. Constructions in Euclidean geometry generally require the use of a compass and ruler, but the new method is very simple, requiring the use of only a ruler. The new construction can also be applied to affine transformations.

Keywords: congruence transformations, similarity transformations, fixed points, Apollonian circles, affine transformations

1. Random dot patterns

In 1982, the author discovered a new method of constructing fixed points in congruence transformations [3]. Constructions in Euclidean geometry generally require the use of a compass and ruler, but the newly developed method required the use of only a ruler. The method was also applicable to the fixed points of similarity transformations. This paper gives a detailed description of the application of the method to similarity transformations.

Assume that two pieces of square, congruent origami paper are placed randomly on a desktop (Fig. 1). It is possible to align these pieces of paper using some combination of translations and rotations (or symmetric displacements), and doing so is referred to as a congruence transformation. It is known, however, that for all congruence transformations there exists at least one point that will not move as a result of the transformation. Such immobile points are called fixed points, and a rotation of a figure using a fixed point as the axis of rotation can be used to align the two figures. In other words, the previous combination of translation and rotation can be replaced by a single rotation.

The construction of a fixed point generally requires the use of a compass and ruler. Taking the vertices of a square as $A, B, C, D,$ and its vertices after transformation as A', B', C', D', the fixed point is constructed by finding the point of intersection formed by the perpendicular bisectors of the line segments AA' and BB'. Doing so is not problematic, but in 1982 I stumbled upon a method for finding these fixed points using only a ruler. The method involves connecting the intersections of opposing sides of the two squares, as indicated in Fig. 1. The point of intersection of the two lines is a fixed point, making this a very simple method of construction of such points. One can place the squares in congruence by pressing down on this fixed point with the point of a compass or the tip of a pen, and then rotating the upper square. I invite readers who have never done so to try this for themselves. If you do not have access to origami paper, then copier paper can be cut into squares to produce the same effect.

I discovered this new construction method when working with random dot patterns like those shown in Fig. 2(1). Copying such a random dot pattern onto an overhead projector sheet, placing the sheet on top of the original pattern, and then rotating it by a small amount produces concentric circles like those shown in Fig. 2(2). The moment at which the concentric circles appear is quite obvious. Placing one's fingertip on the center of the concentric circles and rotating the upper sheet will again bring two patterns into perfect alignment and the circles disappear. In other words, the center of rotation is a fixed point. The random dot patterns used were created with about twenty lines of Visual Basic code, using the built-in RND function to place 2000 points randomly within a square with sides 20-cm in length.

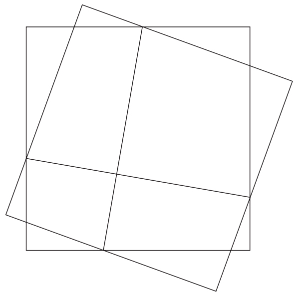
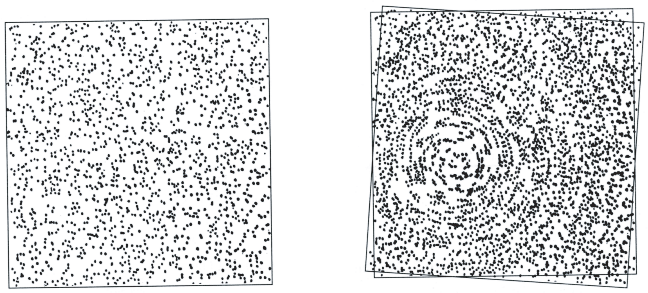


Fig. 1: The fixed point in a congruence transformation



(1) (2)

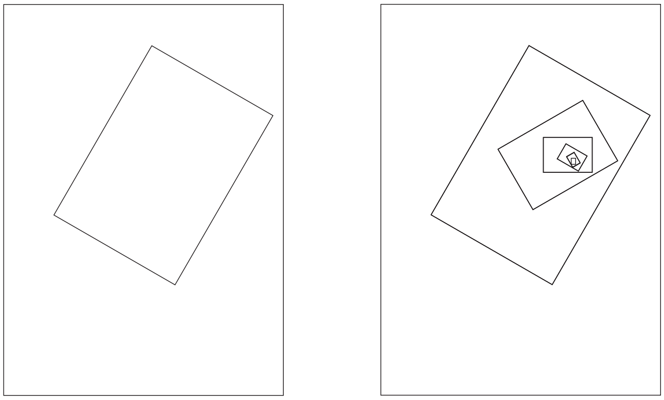
Fig. 2: A random dot pattern

2. Diagrams with differing scales

The main topic of this paper, however, is the fixed points of similarity transformations, not congruence transformations. Figure 3(1) depicts a rectangle, onto which has been placed a similar rectangle (assuming, for the sake of discussion, a scaling factor of 50%). We would now like to find the fixed point for this case. Coxeter's *Introduction to Geometry* gives the example as follows [1]: "If two maps of the same country on different scales are drawn on tracing paper and superposed, there is just one place that is represented by the same spot on both maps. (It is understood that one of the maps may be turned over before it is superposed on the other.)"

This is an example showing the guaranteed existence of a fixed point after a similarity transformation composed of scaling and rotation. Readers who are not familiar with this example should try it by copying a reduced diagram onto a clear overhead projector sheet and placing it on top of the original diagram.

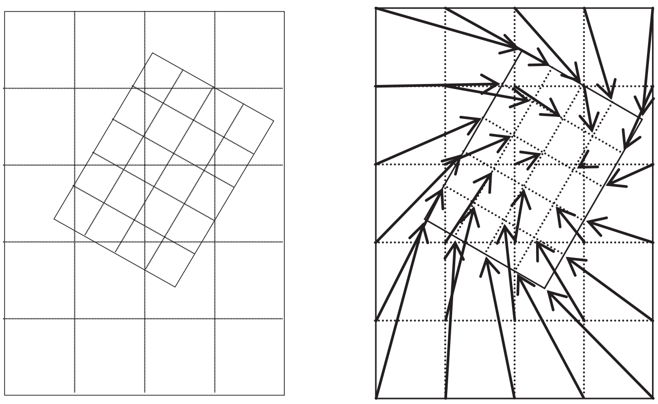
Figure 3(2) shows one method of finding the location of the fixed point. First, the diagram in Fig. 3(1) is copied at a 50% reduction. That copy is then aligned with the smaller rectangle. This reduction copying and alignment process is then repeated multiple times, and the result is that the rectangle will approach the fixed point. The reduced rectangle will of course converge towards the fixed point, but this method is not necessary to determine its location. It can also be constructed using only the relationship between the two rectangles in Fig. 3(1).



(1) (2)

Fig. 3: The fixed point in a similarity transformation

To do so, when taking two similar rectangles we need to consider the similarity transformation not only of the points on their sides, but within the space that they enclose as well. To that



(1) (2)

Fig. 4: Adding arrows between corresponding vertices

end, draw three vertical and four horizontal lines within the rectangles, as shown in Fig. 4(1). Corresponding lines are also created in the reduced rectangles. Next, draw lines connecting the vertices of the original rectangle with the corresponding vertices of the reduced rectangle. The vertices will move as a result of the similarity transformation as indicated by the arrows, but there will be a point that does not move at all. This point is the fixed point of the similarity transformation (Fig. 4(2)).

Performing this operation by hand is time consuming, so I created a program using Visual Basic that allows an increase in the number of vertices. The result is the diagram shown in Fig. 5. The location of the fixed point is clear in the diagram.

3. Construction using a compass and ruler

Figure 5 makes the location of the fixed point clear, and one can also see that there is only a single fixed point. Let us search, however, for a method of precisely locating the position of the fixed point.

Similarity transformations are performed through some combination of three simultaneous transformations: translation, rotation, and scaling. Congruence transformations, however, are performed using only translations and rotations. Figure 6 shows the case where there are no rotations. Here, the relationship between the original rectangle $ABCD$ and the scaled rectangle $A'B'C'D'$ is indicated by the corresponding sides AB and $A'B'$, BC and $B'C'$, CD and $C'D'$, and DA and $D'A'$. Connecting the corresponding vertices results in a single intersection, O, and this point is called the center of similitude, or the homothetic center. The relationship between the scaling from AB to $A'B'$ can be understood from the homothetic center O. The ratio of the lengths of AB and $A'B'$ are called the homothetic ratio.

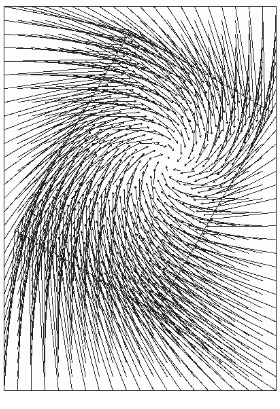


Fig. 5: A computer-generated similarity transformation

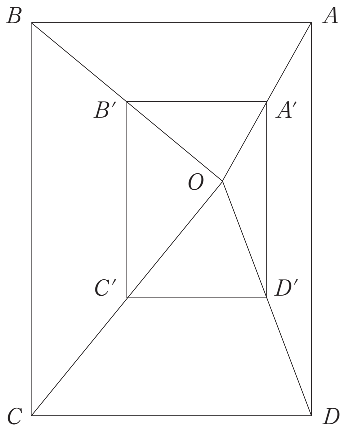


Fig. 6: The homothetic center

The situation indicated by Fig. 6 is very important to the fixed point of a similarity transformation, so the relationship between the positions should be well understood. As we saw in Figs. 3 and 4, the actual similarity transformation adds a rotation to the translation and scaling, so let us consider how to construct the fixed point in such a situation. Generally known methods involve the use of a compass and ruler, and here I will introduce two such methods.

Let P be the point of intersection between sides AB and $A'B'$. In the case where those sides do not intersect, we will consider the intersection created by extension of those sides. Draw the circle formed by the points P, A, and A', and the circle formed by points P, B, and B'. The fixed point will be the point O formed by the intersection of the circles. Figure 7 shows an example construction, and one can see that there is a similarity relationship between ΔOAB and $\Delta OA'B'$, scaling with O as the center and rotating so that side AB moves to side $A'B'$. This construction was performed according to the description in Reference [2].

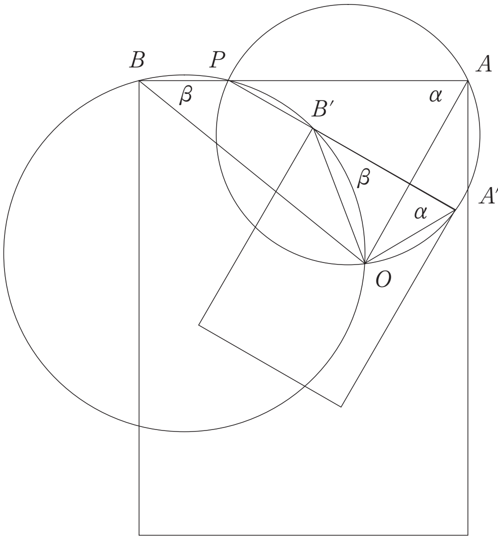


Fig. 7: Using a similarity diagram (Solution 1)

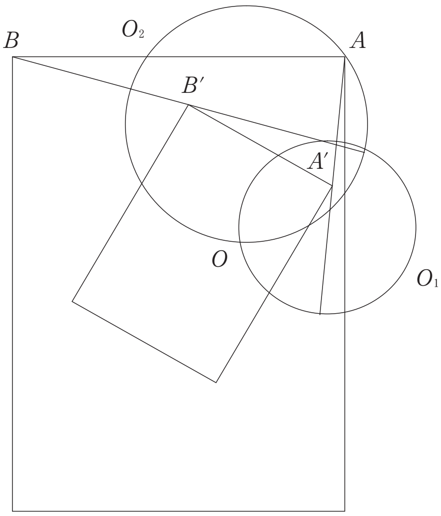


Fig. 8: Using Apollonian circles (Solution 2)

The second construction uses Apollonian circles (Fig. 8). Taking the vertices A and A' on rectangles $ABCD$ and $A'B'C'D'$, the locus of points at a distance of the homothetic ratio ($AB : A'B'$) from those vertices forms an Apollonian circle O_1 with a diameter on the line formed by vertices A and A'. Similarly, the locus of points at a distance of the homothetic ratio from vertices B and B' forms an Apollonian circle O_2 with a diameter on the line formed by those vertices. The intersection O of these two Apollonian circles is the fixed point. Also, $OA : OA' = AB : A'B'$ and $OB : OB' = AB : A'B'$. This construction was performed according to the description in Reference [1].

4. Construction using only a ruler

The above are standard methods of finding the fixed points, but we opened this discussion with the question as to the possibility of applying the method of construction used for congruence transformations to similarity transformations, and indeed this is possible. The method for doing so is shown in Fig. 9.

Theorem 1. *First, find the point of intersection between corresponding sides of rectangle*

ABCD and rectangle A′B′C′D′. In cases where the sides do not intersect, consider extensions of those sides. Take the intersection of sides AB and A′B′ as P, that of sides CD and C′D′ as Q, that of sides DA and D′A′ as R, and that of sides BC and B′C′ as S. The fixed point of the similarity transformation is then the intersection O of the lines PQ and RS. This is the most elegant construction of the point that does not involve the use of a compass.

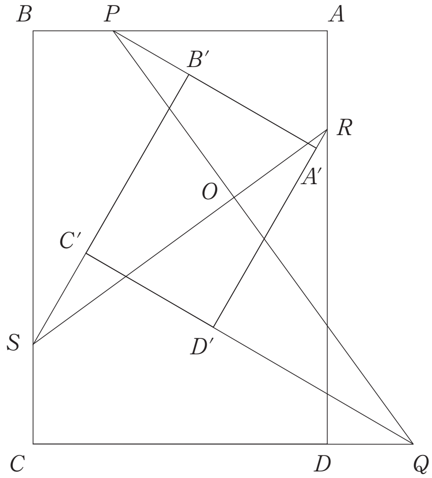


Fig. 9: Construction without using a compass (Solution 3)

Proof. I will now demonstrate why the constructed point O is the fixed point. To do so, it is sufficient to show that the fixed point must lie on both the line PQ and on the line RS. First, assume that the fixed point is some point O on the line PQ. As shown in Fig. 10, the ratio $OH_1 : OH_2$ of the length of the perpendicular segments drawn from point O to the sides AB and $A'B'$ is the homothetic ratio ($BC : B'C'$). Similarly, the ratio $OH_3 : OH_4$ of the length of the perpendicular segments drawn from point O to the sides CD and $C'D'$ is also the homothetic ratio ($BC : B'C'$).

Rotate rectangle $A'B'C'D'$ anti-clockwise about the point O so that the corresponding sides are made parallel, and draw a line from point O to each vertex A', B', C', D'. A reverse calculation of the homothetic ratio on the extended lines gives a scaled rectangle $A''B''C''D''$ of the original rectangle. Comparing the original rectangle $ABCD$ with the rectangle $A''B''C''D''$ that we just constructed, we see that sides AB and $A''B''$ lie on the same line, and sides CD and $C''D''$ lie on the same line (Fig. 11). In other words, taking some point lying on line PQ as the center of rotation, the upper and lower sides are aligned. The corresponding sides will not be brought into congruence as under a congruence transformation, but they are brought into a "superimposed" positional relationship like that of a similarity transformation shown in Fig. 6.

We next perform the same transformation, this time taking some point O on line RS as the center of rotation (Fig. 12). In this case, too, the ratio of the lengths of the perpendicular segments drawn from point O to DA and $D'A'$ is the homothetic ratio. The ratio of the perpendicular segments drawn from point O to BC and $B'C'$ is also the homothetic ratio.

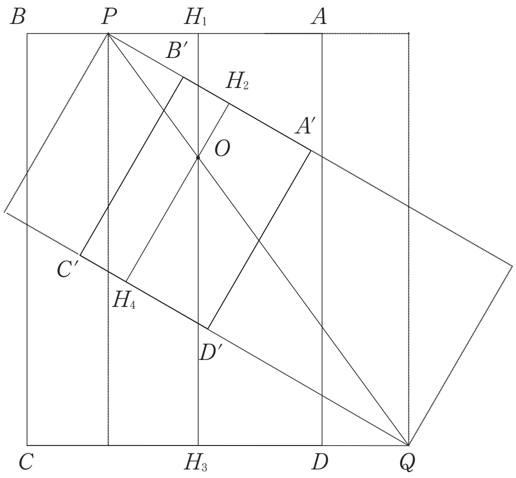


Fig. 10: Taking the center of rotation on the line PQ

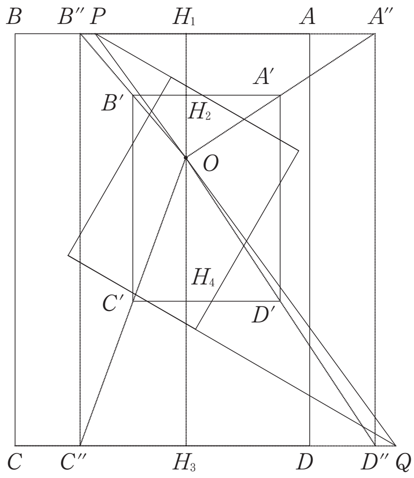


Fig. 11: Aligning the top and bottom sides

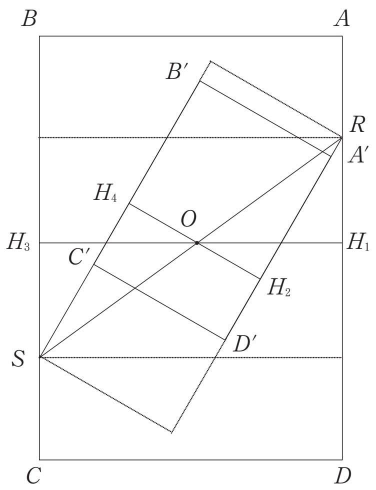


Fig. 12: Taking the center of rotation on the line RS

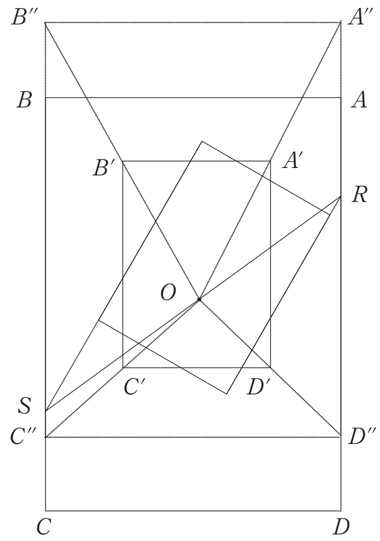


Fig. 13: Aligning the right and left sides

Figure 13 shows rotating the rectangle $A'B'C'D'$ anti-clockwise about the point O so that the four corresponding sides are made parallel. Drawing segments from point O to each of the vertices of the rectangle $A'B'C'D'$, and applying the homothetic ratio to reconstruct the original rectangle results in $A''B''C''D''$. From this diagram, we see that sides $B''C''$ and BC, and sides $D''A''$ and DA lie on the same line. In other words, bringing the point of rotation to line RS aligns the right and left sides.

From Figs. 11 and 13, we see that to align both the top and bottom and the right and left sides, the point of rotation must lie on both line PQ, and moreover on line RS (Fig. 14). In other words, if the intersection O of lines PQ and RS is taken as the point of rotation, then the top and bottom and the right and left sides will align simultaneously. This point of rotation is therefore the fixed point of the similarity transformation.

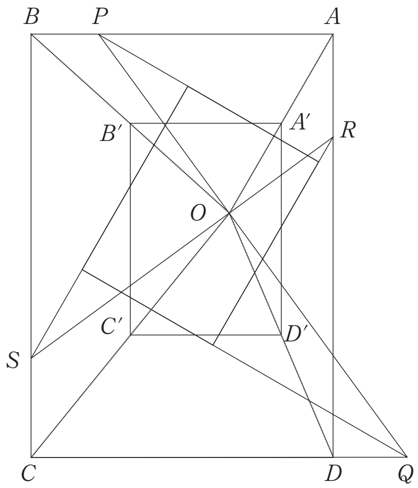


Fig. 14: Aligning the top and bottom and the right and left sides

This paper presents a method of constructing fixed points using only a ruler. The method can be applied not only to congruence transformations and similarity transformations, but also to affine transformations. Because affine transformations are linear transformations by matrices, they allow transformations of both length and angle. Construction of the fixed point described by two arbitrary triangles should also be possible using this method. For details, see Reference [4].

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