The "52 Cycling Cards" Trick Using Modular Arithmetic

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Abstract: Ordering 52 playing cards according to a particular rule allows one to perform a certain magic trick. By learning the rule, after viewing a given card the performer will be able to predict the next card in the deck. The key to the trick is an application of modular arithmetic, one that the author hopes will lend itself as an interesting device for mathematics education.

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1. "Guessing" the cards

A British friend recently sent me an email describing an interesting card trick that he said I should keep secret. The trick uses a standard deck of 52 cards, with each card assigned a numeric value of 1 to 13 and each of the four suits also assigned a numeric value, such as 1 for spades, 2 for hearts, 3 for clubs and 4 for diamonds.

Now for each card, double its numeric value, and add to that the value that was assigned to its suit. If the result of that calculation exceeds 13, divide the value by 13 and use the remainder. This value gives you not only the numeric value for the next card, but also the suit, as follows. If the calculated value was from 1 to 3, then keep the same suit. If it was from 4 to 6, use the other suit of the same color. If it was from 7 to 9, use the previous suit, and if it was from 10 to 13 use the following suit.

My friend's claim was that ordering the cards in this way will allow one to predict the order of every card in the deck. But is such a thing really possible? The claim is that the card values cycle through the entire deck, so I've named it "52 Cycling Cards."

Honestly, I only half believed his claim. I hear of things like this all the time, and some 30 years ago I went through a period where I was very interested in performance magic, so I have some experience with card guessing tricks. Digging through a box in my office I found a collection of old tricks from those days, among which were a deck of "magic cards," as shown in Figure 1. The deck is designed to allow one to "guess" the face value of a card by looking at its reverse. The illustration shows a 9 of diamonds on the left, and a 5 of spades on the right. As you might suppose, the trick lies in the pattern on the back of the cards. I'll leave it to the reader to figure the system out.

Had the trick under discussion relied on the use of a deck such as that shown in Figure 1, my friend's email likely would not have interested me much. A bit of thought showed that it did not in fact rely on such cheap trickery, but on mathematical principles.

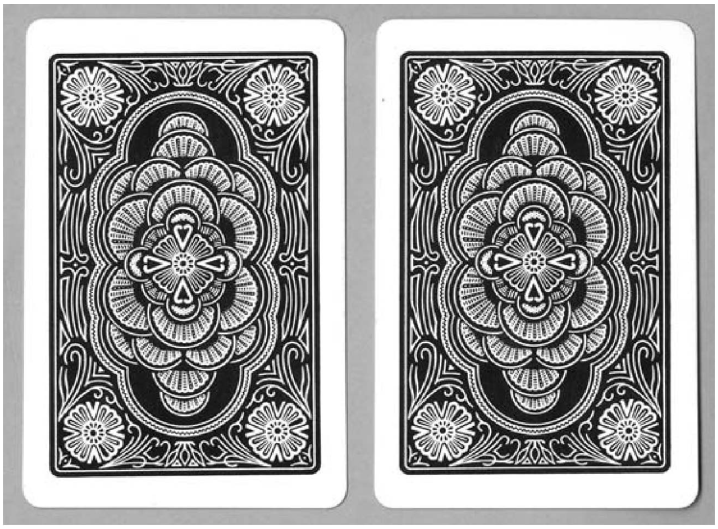

Figure 1. "Magic" cards (left: 9 of diamonds; right: 5 of spades)

2. 52 Cycling Cards

Let us start by reviewing the methodology. The cards each have a numeric value from 1 to 13. When using a standard deck of playing cards, it is convenient to assign the aces a value of 1, and the jacks, queens and kings values of 11, 12 and 13, respectively. There are four suits in two colors, black for spades and clubs, red for hearts and diamonds. For the purposes of our calculations, we will assign the suits the values shown in Table 1, 1 for spades, 2 for hearts, 3 for clubs and 4 for diamonds. Note that it is best to alternate the colors of the suits.

Spades (♠)	1
Hearts (♡)	2
Clubs (♣)	3
Diamonds (◇)	4

Table 1. Numeric assignments for card suits

Let's try guessing the next card, assuming the topmost card to be the 8 of spades (8♠). Doubling the numeric value of the card (8) and adding its suit value (1, since this is a spade), we get

$$8\times 2+1=17$$

There are only 13 card values, yet our calculated value is greater than 13, so we remove multiples of 13 until we get its remainder, 4.

$$17-13=4$$

$$17\equiv 4 \mod 13$$

This is the number of the next card. Representing this as a general expression, where we take n as the card’s numeric value, s as the value associated with its suit, and m as the new number, we get the following:

$$m=n\times 2+s \pmod{13}$$

However, in the case where the result of the calculation is a multiple of 13, the remainder will be 0, so in that case we will take the result to be 13. We next want to compare this new number m to Table 2 to determine the suit. If the calculated value was from 1 to 3, then the suit does not change. If it was from 4 to 6, it will be the other suit of the same color, if it was from 7 to 9, it will be the previous suit in our ordering, and if it was from 10 to 13 it will be the following suit. The value of m that we found was 4, which according to Table 2 indicates the other suit of the same color. Spades (♠) are black, so the same-colored other suit is clubs (♣). So the next card will be 4♣.

Face value	Suit
1-3	Same suit
4-6	Same color, different suit
7-9	Earlier suit
10-13	Following suit

Table 2. Correspondence between values and suits

For practice, let us calculate the card that would follow 4♣. The numeric value for clubs is 3, so our new number is

$$4\times 2+3=11$$

Since 11 is in the range 10-13, the suit will be the following one, diamonds (♢). The next card, therefore, will be 11♢.

So now we understand the methodology by which we determine the sequence of the cards. What I still was not convinced of, however, was whether following this methodology would allow us to order all 52 cards. I used a spreadsheet program to confirm that my friend’s method worked for each case. I was thrilled to see that yes, indeed, no matter what card I started with I would end up with a full sequence to follow it.

For reference, Figure 2 shows that each of the 52 cards is used without repetition. You can read that table as follows. The column on the left lists the cards from 1 (A) to 13 (K), and the header at top shows each of the suits from spades (♠) to diamonds (◇), along with their corresponding numeric values 1 through 4.

To find the card that should follow A♠, look at the intersection of the corresponding row and column, which indicates that the following card should be 3♠. Looking in turn at the intersection between the 3 row and the ♠ column indicates that the card after that should be 7◇. I invite the reader to perform the calculations for the next few cards and confirm that they follow the entries in the Figure 2.

You can create your own table using a spreadsheet, but because 52 calculations are involved there is always the risk of mistyping or otherwise entering the wrong calculation into a cell. An alternative method is to write a short program to do the calculations, something that should only take 20 lines or so using a language such as Visual BASIC. I created such a program myself, and the results form the content for Figure 2.

	1	2	3	4
	♠	♡	♣	◇
A	3♠	4◇	5♠	6♡
2	5♣	6◇	7♡	8♣
3	7◇	8♠	9♡	10♠
4	9◇	10♣	J◇	Q♠
5	J♡	Q♣	K◇	A◇
6	K♡	A♡	2♣	3◇
7	2♠	3♡	4♠	5♡
8	4♣	5◇	6♠	7♣
9	6♣	7♠	8♡	9♣
10	8◇	9♠	10◇	J♠
J	10♡	J♣	Q◇	K♠
Q	Q♡	K♣	A♣	2◇
K	A♠	2♡	3♣	4♡

Figure 2. The 52 cycling cards

Figure 2 contains a lot of information, making it somewhat difficult to understand. I therefore also created Figure 3, which contains only the numeric values for each card. As we discussed before, the value for the next card number m is found by doubling the current card value n and adding to that its suit value s. Should that number exceed 13, we instead take the remainder after removing multiples of 13. We can see that each of the numbers 1 through 13 appears four times in the table, and so, numerically, at least, there does not seem to be a problem.

Figure 4 shows the correspondence between the new number m and its suit s. Recall that

	1	2	3	4
	♠	♡	♣	◇
1	3	4	5	6
2	5	6	7	8
3	7	8	9	10
4	9	10	11	12
5	11	12	13	1
6	13	1	2	3
7	2	3	4	5
8	4	5	6	7
9	6	7	8	9
10	8	9	10	11
11	10	11	12	13
12	12	13	1	2
13	1	2	3	4

Figure 3. Cycling numbers

if the calculated value was from 1 to 3, then the suit does not change. If it was from 4 to 6, it will be the other suit of the same color, if it was from 7 to 9, it will be the previous suit, and if it was from 10 to 13 it will be the following suit. The randomness is increased by the pairings between card numeric values and suits.

	1	2	3	4
	♠	♡	♣	◇
1	3	4	5	6
2	5	6	7	8
3	7	8	9	10
4	9	10	11	12
5	11	12	13	1
6	13	1	2	3
7	2	3	4	5
8	4	5	6	7
9	6	7	8	9
10	8	9	10	11
11	10	11	12	13
12	12	13	1	2
13	1	2	3	4

Figure 4. Correspondence with suits.

3. Randomness from 1 to 52

Since there are 52 cards in a standard deck, let us take a different approach and consider ways in which to generate a random sequence from 1 to 52.

It is generally possible to generate a pseudo-random sequence using prime numbers and primitive roots. For example, using the relationship between the prime number 7 and the primitive root 3 we can generate the following random ordering of the numbers from 1 to 6:

$3^1 = 3$

$3^2 = 3^1 \times 3 \equiv 9 \equiv 2 \mod 7$

$3^3 = 3^2 \times 3 \equiv 2 \times 3 \equiv 6$

$3^4 = 3^3 \times 3 \equiv 6 \times 3 \equiv 18 \equiv 4 \mod 7$

$3^5 = 3^4 \times 3 \equiv 4 \times 3 \equiv 12 \equiv 5 \mod 7$

$3^6 = 3^5 \times 3 \equiv 5 \times 3 \equiv 15 \equiv 1 \mod 7$

Here we have used powers of the primitive root 3, removing powers of 7 when the result is greater than $7 - 1 = 6$. This results in the sequence

3, 2, 6, 4, 5, 1

Another way of expressing this is to say that 3^1, 3^2, 3^3, 3^4, 3^5, 3^6 is a cyclic group of modulo 7. It also upholds two important properties of randomness: "independence," which means that a given number in the sequence does not affect the values preceding or following it, and "uniformity," which means that the entire group is covered.

A general relationship between a prime number p and a relatively prime number r is as follows:

$r^{p-1} \equiv 1 \mod p$

This relationship is known as Fermat's Little Theorem. It requires a prime number p, because only prime numbers can create a random string of period $p-1$. [1]

Just as 3 is one of the primitive roots of the prime number 7, given a primitive root of the prime number $p=53$, we can create a sequence of random numbers from 1 to 52. Based on that, I wrote another Visual BASIC program to find the primitive roots from 1 to 52 for the prime number $p=53$. That program gave me the following:

2, 3, 5, 8, 12, 14, 18, 19, 20, 21, 22, 26, 27, 31, 32, 33, 34, 35, 39, 41, 45, 48, 50, 51

You might find it interesting to select one of these numbers and confirm that after 53 calculations you will return to the number you started from.

Using 53 as our prime we have 24 primitive roots, approximately 45% of the numbers from 1 to 53. I was surprised by how many primitive roots there were. Did the prime number 53 have an extraordinarily large number of them? Investigating the 25 prime numbers between 1 and 100, I found that 83 had the largest percentage of primitive roots at 48% (40 primitive roots), while 31 had the smallest at 26% (8 primitive roots). The average is therefore 37%, so the prime 53 does not seem to be particularly special. I am curious whether there might be some kind of approximating function that describes the number of primitive roots r for a prime number p as p approaches infinity, in the same way that the prime number theorem describes the distribution of primes.

Here we have discussed random number sequences of the numbers 1 to 52, and the possibility of applying this principle to card tricks. Unfortunately the calculations involved are somewhat complex for mental arithmetic, making it impractical for use in stage magic.

4. Why are there 52 cards in a deck?

The most commonly used type of playing cards worldwide has 52 cards in a deck. I wondered if the same values and suits had always been used with this type of deck.

There are a number of theories as to the origins of playing cards, but the common thread between them is that they originated in Asia and crossed over to Europe, where they developed into their modern form. Playing cards similar to those used today appeared throughout Europe in the latter half of the 14th century, and later went through changes in design, shape, count, and naming in various regions. Some say that the spades were originally swords, indicating the military or nobility, hearts were goblets, indicating the priesthood, diamonds were currency, indicating merchants, and clubs were, well, clubs, indicating the peasantry. The deck that is now standard throughout the world was developed in England and the United States between the 19th and 20th centuries, and became predominant due to the power of those nations at the time.

There are several theories why there are 52 cards. For example,

(1) The four suits represent the four seasons, each season having 13 weeks ($13\times4=52$).

(2) The sum of the numbers 1 through 13 is 91, which is the number of days per season. Multiplying this by 4 gives 364 days in a year. Since this is one day short, the joker was introduced. The second joker is for the extra day in a leap year.

(3) The number of letters used in the English spelling of the ranks (ace, two, three, ..., king) totals 52.

Setting theory (3) aside, the above seems to be a plausible explanation for the existence of the jokers. Nonetheless, a 1983 book by Olney Richmond indicates that this is merely folklore with no evidence behind it (see the "Playing Card" entry on Wikipedia).

Continuing on, I next wondered if there might be some other rule for creating 52 cycling cards. Using the Visual BASIC program, I tried changing the correspondence table for formulas and suits to look for such a rule. Doing so presented this one:

Given a current value n, and a suit number s, the new value m will be

$$m=n\times3+s \pmod{13}$$

In the first equation we doubled the face value, but here we are tripling it. To determine the suit, for values 7 through 9 we keep the same suit, for 10 through 12 we use the other suit of the same color, for values of 4 through 6 we use the preceding suit, and for values 1 through 3 and 13 we use the following suit（Table 3）.

Face value	Suit
7-9	Same suit
10-12	Same color, different suit
4-6	Earlier suit
1-3, 13	Following suit

Table 3. Correspondence between values and suits（second method）

I found other methods for creating 52 cycling cards, but the values used in the calculations were too large for mental arithmetic, and the correspondences with the suits were too complex. Needless to say, the first method under discussion here was the simplest one with the easiest calculations.

Nonetheless, discovering that there exist multiple methods to the same end was an interesting discovery for me. I notified by British friend that additional solutions existed, and he was very happy to learn this. He asked me to keep the method secret, but I'm sure that he would not mind it being published in a mathematics academic journal. I hope that the reader will try this out with a deck of cards at hand.

References

[1] Y. Nishiyama, Opening the black box of random numbers, *Sugaku wo Tanoshimu*（Mathematics in Daily Life），(2007)，182-189. Gendai Sugakusha, Kyoto,（in Japanese）.