Mathematics Column (17)

Exchanging Gifts

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There is a form of random gift exchange that some people use for Christmas parties and the like, in which everyone in a group brings a present, and those presents are randomly distributed through drawing lots. The problem with this is that there's a possibility of participants drawing their own gift, defeating the purpose of the gift exchange. It's a breach of etiquette to identify one's own gift, but disappointing to lose the surprise of what one will receive.

A similar thing can happen when reorganizing seating charts. I don't know if they still do so these days, but when I was in elementary and junior high school we often had new seating assignments at the start of each semester. Apparently, this was done to promote communication with all of one's classmates, and it was something we always looked forward to.

Randomly rearranging seats, however, often resulted in some student remaining where they were. This made that student the focus of the class's attention as the teacher told them to swap with someone else, an embarrassing experience when one is young.

Such accidents in gift exchange and seating reassignment happen with great frequency. In 1708, P.R. Montmort proved that we can calculate the probability of such an event as

$$\mathrm{P} = 1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \cdots \pm \frac{1}{n!},$$

where n is the number of participants (Feller, 1957). Also known is that as n approaches infinity, this value approaches

$$1 - \frac{1}{e} \quad (\approx 0.63212),$$

where $e \approx 2.71828$ is the base of the natural logarithm.

It is interesting that regardless of the number of people involved—whether you have 100 people in the gift exchange or 1000 students switching seats—there is an approximately two-thirds chance that someone will receive their own gift or be stuck in their old seat.

Let's perform some simple calculations. Say you have two children, child_1 and child_2, with two presents, present_1 and present_2.

In this case there are 2! = 2 ways to distribute the presents, only one of which results in child_1 getting present_2 and child_2 receiving present_1, so the probability of drawing one's own present is 1 ÷ 2 = 0.5.

When there are three children with three presents, there are 3! = 6 possible distributions, 4 of which result in at least one self-gifting child, so the probability is 4 ÷ 6 = 0.67. With four children there are 4! = 24 distributions, which include 15 self-gifters, so the probability is 15 ÷ 24 = 0.625.

So with 2, 3, and 4 children we get respective probabilities of 0.5, 0.67, and 0.625, which does put us in the area of 0.6. Feel free to calculate the probabilities for more children, but the counting quickly become tedious, so I'll stop at four and stick to Montmort's equation beyond that. His derivation of the formula involves set arithmetic using Venn diagrams and so may be somewhat challenging, but you should be able to follow the argument if you take things slow, something I highly encourage you to do.

References

Feller, W. (1957). *An introduction to probability theory and its applications*. New York: Wiley.