Mathematics Column (20)

Humble's Prediction

Yutaka Nishiyama

My friend S. Humble is quite enthusiastic about mathematics education. He has developed a mathematical puzzle for elementary school students, introduced in the *New York Times* (13 May 2013) as "Triangle Mysteries," that has deep connections with modern mathematics.

I am quite captivated by this puzzle of his. It goes like this: Prepare a deck of cards in three colors (we'll use red, blue, and yellow, designated as R, B, and Y, respectively). Shuffle the deck and reveal four cards in a row. Let's say that we dealt out

R, Y, B, B.

We next add a row of three cards below this, following two rules. (1) If there are two adjacent cards of the same color, we place a card of that color below them. (2) If there are two adjacent cards with differing colors, we place a card of the third color below them.

So let's see what happens. The first pair, R and Y, are of different colors, so we place a B card below them. Similarly, we place an R card below the Y–B pair. The last two are of the same color, so we place another B card below them. Continuing on for two more rows, we end up with this:

```
R   Y   B   B
  B   R   B
    Y   Y
      Y
```

What Humble discovered while playing this game with children is that we can easily predict the color of the final card from the first row alone. In the above case, the leftmost and rightmost cards in the first row are R and B, so by applying the rules we can predict that the final card will be Y.

I'll admit that I was dubious about this claim when Humble first told me about it. Upon exhaustively investigating each of the $3^4 = 81$ possible starting rows, however, I found that the trick indeed worked every time. I therefore named this Humble's Prediction.

```
R   Y   B   B
  B   R   B
    Y   Y
      Y
```

I love puzzles, so I tried this with initial rows of more than four cards. It didn't work when starting with rows of five or six. Humble found through teaching that it works for starting rows of ten cards. With ten cards there are $3^{10} = 59049$ possible staring positions, so if you would like to perform a brute force verification, you'll want to use a computer, something I highly recommend you attempt.

Since we're using two colors to determine a third, this is a sort of binary operation. Furthermore, we're operating on a set of three elements (colors) that are closed under this operator, so they form a group. The order of elements in the operation does not matter, so this is an abelian group.

Also interesting is that the final color is determined by repeated application of a binary operator, so there is some relation with Pascal's triangle. The version of Pascal's triangle that we learn in high school is something like that on the left, below, but if we instead display it as remainders after division by 3 we get the triangle on the right.

```
        1                   1
      1   1               1   1
    1   2   1           1   2   1
  1   3   3   1       1   0   0   1
1   4   6   4   1   1   1   0   1   1
```

We obtain rows with leftmost and rightmost values of 1 and all other values of 0 in rows 4, 10, 28, ⋯. In other words, Humble's Prediction holds when the number of cards n in the initial row is

$$n = 3^s + 1, (s = 0, 1, \cdots).$$

So while we started with a game designed as educational material for elementary students, we've ended up with some relatively advanced mathematics. I urge readers who are intrigued by this puzzle to investigate how the above equation was derived.