Mathematics Column (24)

$\mathbf{1+2+3+4+\cdots=-\frac{1}{12}}$

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I found this very curious equation on page 111 of Oguri Hiroshi's *Ōguri-sensei no chōgen-riron nyumon* [Oguri's Introduction to Superstring Theory] (Kodansha Blue Backs). The left side is monotonically increasing, and so should diverge to infinity. The right side, however, is a negative value—nothing like infinity by a long shot. An Internet search turned up a similar equation:

$$1+8+27+64+\cdots=\frac{1}{120}.$$

This equation is used in superstring theory, in relation to a physical phenomenon called the Casimir effect, demonstrated by Lamoreaux et al. in 1997.

As is well known, the harmonic series diverge to infinity.

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots=\infty$$

This equation makes an appearance in the book-stacking problem, where one investigates the maximal overhang of blocks stacked on the edge of a table. This is obtained by shifting the stacked blocks, from the bottom up, by one-half a block width, then one-third, then one-fourth, and so on; doing so allows an arbitrarily large overhang. Divergence to infinity occurs at a logarithmic rate, and so is quite slow, but theoretically you could create an infinite overhang.

Also famous is the Basel problem, solved by Euler in 1735. When you square the denominators of each term in the harmonic series, the resulting series converges to $\frac{\pi^2}{6}$:

$$1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots=\frac{\pi^2}{6}.$$

The above is on the order of what's learned in the first couple of years of undergraduate mathematics studies, and so well within reason. In later studies, students encounter something new, the zeta function, which takes *s* as a complex parameter for the function $\zeta(s)$:

$$\zeta(s)=1+\frac{1}{2^s}+\frac{1}{3^s}+\frac{1}{4^s}+\cdots$$

Within the scope of real numbers, if $s>1$ the zeta function gives a finite value, and if $s\le 1$ the series diverges. The function was thus initially defined for the domain $s>1$, but Euler used the following method to expand the domain of $\zeta(s)$.

The geometric series with first term 1 and infinite common ratio *x* is given by the following formula:

$$1+x+x^2+x^3+\cdots=\frac{1}{1-x}. \qquad (1)$$

This equation holds when $|x|<1$, but not when $|x|\ge 1$. But Euler considered the following: Indeed, if $x=1$ then substitution gives a denominator of 0 on the right, resulting in an infinite value. But if we instead substitute $x=-1$, we get the following pseudo equation:

$$1-1+1-1+\cdots=\frac{1}{2} \qquad (2)$$

Taking the derivative of Eq. (1) gives Eq. (3),

$$1+2x+3x^2+4x^3+\cdots=\frac{1}{(1-x)^2}, \qquad (3)$$

and substituting $x=-1$ gives

$$1-2+3-4+\cdots=\frac{1}{4}. \qquad (4)$$

From Eq. (4) and

$$\begin{aligned}&1-2+3-4+\cdots\\&=(1+2+3+4+\cdots)-2\times(2+4+\cdots)\\&=-3\times(1+2+3+4+\cdots),\end{aligned}$$

we can derive

$$\zeta(-1)=1+2+3+4+\cdots=-\frac{1}{12},$$

as presented at the beginning. This method is called analytic continuation.

This method for extending the domain of the function that Euler was invented was later established by Riemann as the zeta function. From this, the zeta function $\zeta(s)$ is defined in such a way that it gives a finite value for all values of *s* other than $s=1$.

Methods for extending functions like this are similar to the development of non-Euclidean geometry. Denying the fifth postulate of Euclidean geometry about parallel lines resulted in the birth of non-Euclidean geometry. Considering that both geometries can simultaneously exist without contradiction makes this expansion of the zeta function's domain by analytic continuation less mysterious.