J. RECREATIONAL MATHEMATICS, Vol. 8(2), 1975-76

REPDIGIT TRIANGULAR NUMBERS

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A triangular number is a positive integer of the form $T_n = n(n+1)/2$ where n is a positive integer.

In 1905, E. B. Escott [1] proved that 1, 3, 6, 55, 66 and 666 are the only triangular numbers of less than 30 digits that consist of a single repeated digit. This paper will complete the proof of this theorem and show that there are no other triangular numbers of any digit length which consist only of a single repeated digit. Formally, we have the following theorem.

Theorem: The triangular numbers 1, 3, 6, 55, 66 and 666 are the only triangular numbers consisting of a single digit, alone or repeated.

Proof: If a triangular number, T_k, is to consist of $j-1$ like digits, we must have

$$T_k = \frac{k(k+1)}{2} = d(10^j - 1)/9. \tag{1}$$

Solving for k, we have

$$k = [-9 \pm (81 + 72d(10^j - 1))^{\frac{1}{2}}]/18 = [-1 \pm (1 + 8d(10^j - 1)/9)^{\frac{1}{2}}]/2. \tag{2}$$

For k to be an integer, it is necessary that

$$n = 1 + 8d(10^j - 1)/9 \tag{3}$$

be a perfect square. As can be seen from the following table:

$n \pmod{10}$	$n^2 \pmod{10}$	$\frac{n^2+n}{2} \pmod{10}$	$n \pmod{10}$	$n^2 \pmod{10}$	$\frac{n^2+n}{2} \pmod{10}$
0	0	0	5	5	5
1	1	1	6	6	6
2	4	3	7	9	8
3	9	6	8	4	6
4	6	5	9	1	5

Then 2, 4, 7, 9 cannot terminate a triangular number. Consequently $d = 1, 3, 5,$ 6 or 8.

If $d = 1$, from equation (3) then n is 9 or a number of the form $88\cdots889$. If $n \neq 9$, then from equations (1) and (2), $k = 1$ and $T_1 = 1$.

Assume $88\cdots89$ is to be a square, say z^2. Then in decimal notation,

$$z = d_n d_{n-1} \cdots d_2 d_1 d_0.$$

Since $88\cdots89$ end in 9, d_0 must be 3 or 7. If d_0 is 3, then to get the last two digits of $88\cdots89$ to be 89, d_1 must be 3 or 8. If d_1 is to be 8, then from the form of $88\cdots89$, d_2 must be 5. Now there is no choice of d_3 such that $(d_3\,583)^2$ has its last four digits of form 8889. Thus this chain stops. Now d_1 was 3 or 8 and the choice of 8 lead to a contradiction, so we keep $d_0 = 3$ and choose $d_1 = 3$. Then d_2 must be 8 which again leads to a contradiction; we have thus eliminated all possibilities where d_0 was 3. But there were two choices for d_0 so we choose $d_0 = 7$ and begin the same analysis.

We represent the procedure just outlined with the following notation

3 - 8 - 5 - *.

The 3 is the initial choice for d_0; the digit 8 is one of the possibilities for d_1; 5 is a possibility for d_2; * means that there is no acceptable digit for d_3 so as to make the last four digits of $(d_3\,583)^2$ of the correct form.

Thus to obtain a number of the form $88\cdots89 = z^2$, the possibilities for 2 are:

3 - 3 - 8 - *
3 - 8 - 5 - *
7 - 1 - 9 - 7 - *
7 - 1 - 9 - 2 - 2 - *
7 - 1 - 9 - 7 - 5 - *
7 - 1 - 4 - *
7 - 6 - 1 - *
7 - 6 - 6 - 1 - *
7 - 6 - 6 - 6 - 1 - *
7 - 6 - 6 - . . . - 1 - *.

Thus all possibilities are eliminated and no number of the form $88\cdots889$ can be a perfect square. Therefore $T_1 = 1$ is the only triangular number having no digits other than 1.

If $d = 3$, by (3) then n is 25 or a number of the form $266\cdots665$. But any square number terminating in 5 ends in 25. Hence the only T_k consisting of 3's is $T_2 = 3$.

If $d = 5$, from (3) then n has the form $44\cdots441$. Now $441 = (21)^2$, and the possibilities for more than two 4's are:

1 - 2 - 5 - *
1 - 7 - 2 - *
1 - 7 - 7 - 5 - 2 - *
1 - 7 - 7 - 5 - 7 - *

9 - 2 - 2 - 4 - 2 - 3 - 5 - *
9 - 2 - 2 - 4 - 2 - 8 - *
9 - 2 - 2 - 4 - 7 - *
9 - 2 - 2 - 9 - *
9 - 2 - 7 - *
9 - 7 - 4 - *
9 - 7 - 9 - 2 - *
9 - 7 - 9 - 7 - 6 - *
9 - 7 - 9 - 7 - 1 - 4 - 1 - 6 - *
9 - 7 - 9 - 7 - 1 - 4 - 6 - *
9 - 7 - 9 - 7 - 1 - 9 - *.

Thus, the only triangular number having all digits 5 is $T_{10} = 55$.

If $d = 6$, from (3) then n is 49, 529, or a number of the form 533· · ·329.

If 555· · ·329 is to be a perfect square, it must be 529 or 5329. The other possibilities are:

3 - 2 - 3 - *
3 - 2 - 8 - 1 - *
3 - 2 - 8 - 6 - 3 - 4 - *
3 - 2 - 8 - 6 - 3 - 9 - 1 - *
3 - 2 - 8 - 6 - 3 - 9 - 6 - 2 - *
3 - 2 - 8 - 6 - 3 - 9 - 6 - 7 - 1 - 4 - *
3 - 2 - 8 - 6 - 9 - 6 - 7 - 1 - 9 - *
3 - 2 - 8 - 6 - 3 - 9 - 6 - 7 - 6 - *
3 - 2 - 8 - 6 - 8 - *
3 - 7 - 5 - *
7 - 2 - 4 - *
7 - 2 - 9 - 1 - 3 - 5 - *
7 - 2 - 9 - 1 - 8 - *
7 - 2 - 9 - 6 - *
7 - 7 - 1 - 3 - 1 - *
7 - 7 - 1 - 3 - 6 - 5 - *
7 - 7 - 1 - 8 - *
7 - 7 - 6 - *

Hence the only triangular numbers composed only of 6's are $T_3 = 6$, $T_{11} = 66$, and $T_{36} = 666$.

If $d = 8$, from (3) then n is 705 or a number of the form 711· · ·105. But the penultimate digit of any square ending in 5 must be 2. Hence no T_k exists consisting only of 8's.

Thus the theorem is proven.

Reference

1. E. B. Escott, "Math. Quest. Educ. Times," Vol. 8, pp. 33-34, published by C. F. Hodgson & Son, London, 1905.

J. RECREATIONAL MATHEMATICS, Vol. 8(2), 1975-76

COFACTORS OF REPUNITS

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Introduction

Consistent with established usage, a *primitive divisor* of $a^n - b^n, a > b$, is a natural number which divides $a^n - b^n$ but does not divide $a^m - b^m$, $n > m$. Birkhoff and Vandiver [1] referred to divisors which are not primitive as *imprimitive*, but Brillhart and Selfridge [2] used the term *algebraic*. The factorization of numbers of the form $10^n - 1$, a special case of $a^n - b^n$, has been of interest for at least seven centuries [3], and is of marked interest today [4]. One reason for this attention, aside from the challenge that special form factoring holds for many, is that the *period length* of each prime divisor of $10^n - 1$ is n, and no other primes have a period length of n. For a discussion of period lengths and their relationship to repunits, see [5]. A period length is the length of the repeating set of digits in the decimal evaluation of the reciprocal of the prime. For example, the prime 7 has a period length of 6: $\frac{1}{7} = 0.\underline{142857}142857\ldots$

Because $10^n - 1$, written as a string of n 9's, has the algebraic divisor 9, whenever $n > 1$ (and is therefore always composite), it is more convenient to work with the *repunit* $R_n = (10^n - 1)/9$, which is written as a string of n 1's and may or may not be composite. We adapt the terms primitive divisor, algebraic divisor, and cofactor, used by Brillhart and Selfridge [2], to repunits. A *primitive divisor of repunit* R_n is one which divides R_n but divides no smaller repunit. An *algebraic divisor of repunit* R_n is one which divides R_n as well as some smaller repunit. The *primitive cofactor* P_n of R_n is the product of all primitive prime divisors of R_n. The *algebraic cofactor* A_n of R_n is the quotient R_n/P_n.

In this paper we discuss cofactors of repunits, exhibit tables of primitive and algebraic cofactors of R_n, $1 \leq n \leq 100$, and present a method of formulation of algebraic cofactors which supersedes an entirely different method given in an earlier paper by the author [6]. The previous paper contained an error which caused incorrect results in a few cases, and the method did not provide for all possible situations. The approach given here is simpler and complete, and has been checked more thoroughly to make sure that it is correct in all cases.