# Coin Sequence Probabilities and Paradoxes

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1. A COIN is tossed repeatedly, producing an endless random sequence of heads and tails (denoted hereafter by $H$ and $T$). Players $A$ and $B$ wager on the first occurrence of their own nominated sub-sequence. Suppose $A$ chooses $HH$ and $B$ chooses $TH$. $A$ wins if the random sequence above produces $H, H$ in consecutive places before it produces $T, H$ in consecutive places in that order. As the probability of each sub-sequence is $\frac{1}{4}$, it might be thought that the wager was fair; but look at things this way.

$A$ wins if the first two outcomes produce $HH$, and the probability of this is $\frac{1}{4}$.

$B$ wins in all other cases; for as soon as any $T$ occurs, $TH$ must occur before any $HH$.

Therefore the odds in favour of $B$ are $3:1$.

2. Extending to sub-sequences of length 3, suppose $A$ chooses $TTT$. $B$ would then do better with any of $HHT$, $HTH$, $THH$, $THT$; but $HTT$ is best of all having odds of $7:1$ in its favour. If we now look at every possibility for $A$, and at its best beater for $B$, we get the following figure (Fig. 1).

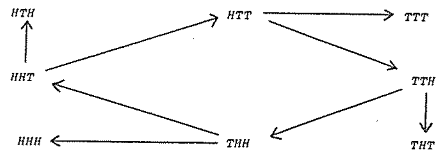


Fig. 1

Thus we get an intransitive ring, but with "fly offs" producing something like a Catherine wheel. It follows that there is no best sub-sequence for $A$ if $B$ chooses second knowing $A$'s choice. But if $B$ chooses completely at random, and $A$ knows this is going to happen, he can maximise his own probability of success by selecting $HTT$ or $THH$.

3. Before proceeding further, we should explain how to evaluate odds in general. These are done in terms of Conway Numbers (see reference 1) which are defined below. We shall work in terms of the particular example

$$\begin{array}{l} A-HTHH \\ B-THTH \end{array}$$

The answers we require are expressed in terms of component figures denoted by $AA$, $AB$, $BA$, $BB$. To get the "$AA$" term, we write down $A$'s sub-sequence underneath itself getting

$$\begin{array}{l} 1 \\ A-HTHH \\ A-HTHH \end{array}$$

We put the digit 1 above the first $H$ because the $H$ leads a sequence which is identically the same as the four terms immediately below it. To get the next digit we discard this $H$ and slide the remaining three letters along to get

$$\begin{array}{l} 0 \\ THH \\ HTHH \end{array}$$

This time the digit is 0 because the $T$ leads a sequence which is not the same as the *three* terms immediately below it.

Continuing with this process we get

$$\begin{array}{ll} 0 & 1 \\ HH & H \\ HTHH & HTHH \end{array}$$

Combining these results into a single display gives

$$\begin{array}{l} 1\,0\,0\,1 \\ HTHH \\ HTHH \end{array}$$

We now interpret 1001 as a four digit number in the binary scale giving 9, and this is called the Conway Number corresponding to $AA$.

Doing this for all possible pairings of $A$ and $B$ gives

$$\begin{array}{ll} 1\,0\,0\,1=9 & 0\,0\,0\,0=0 \\ A-HTHH & A-HTHH \\ A-HTHH & B-THTH \\ 1\,0\,1\,0=10 & 0\,1\,0\,1=5 \\ B-THTH & B-THTH \\ B-THTH & A-HTHH \end{array}$$

For convenience we shall write

$$\begin{array}{ll} AA=9, & AB=0, \\ BB=10, & BA=5. \end{array}$$

It is now the case (not mentioned in the Martin Gardner article[1]) that the average waiting time to get $A$ from scratch is $2.AA = 18$. Also on average the further number of tosses required to complete or get an $A$ having just completed a $B$ is $2.AA-2.BA = 8$. It is however mentioned that the odds in favour of $B$ winning the wager are $(AA-AB):(BB-BA)$.

Applying these results to our particular case,

> $A$ has an average waiting time of 18 to get $HTHH$,
> $B$ has an average waiting time of 20 to get his $THTH$

implying an advantage to $A$. Yet the odds in favour of $B$'s wager are

$$\frac{9-0}{10-5}=\frac{9}{5}.$$

By now, this is a paradox within a paradox.

**4.** Before proving these results ourselves, we shall generalise the situation. Tossing a penny is like rolling a 2-sided die. We shall consider what happens on rolling a $k$-sided die.

**Theorem X**

The expected waiting time to get a given sequence of integers from the set $\{1, 2, \ldots, k\}$ is $k.AA$, where $AA$ stands for the corresponding Conway Number consisting entirely of 0's and 1's and interpreted in the scale of $k$.

**Theorem Y**

Given a sequence $B$ to start with, the expected further number of rolls required to complete or produce the $A$ sequence is $k.AA - k.BA$.

**Proofs**

We proceed by induction, supposing the results to be true for all pairs of sequences $A, B$ of length $\leqslant n$. We then establish the results for length $n+1$.

(*i*) Let $A, B$ both have length $n+1$. If they are identical, the $A$ sequence is complete by virtue of $B$ itself, so the further number of rolls required is zero. This fits in with theorem $Y$ which says that the expected further number required is $k.AA - k.BA = k.AA - k.AA = 0$.

(*ii*) If they are not identical, $BA$ must begin with a zero. Suppose the first $r$ digits of $BA$ are all zero, and that the $(r+1)$th is 1. Then $B$ consists of a "zeros-generating" sub-sequence $C$ followed by a sub-sequence $D$ generating a leading 1. Because of this, $D$ is a leading sub-sequence of $A$.

(*iii*) By inspection

$BA = DA$
$\quad = DD$ as only the first $n+1-r$ terms of $A$ are involved.

(*iv*) To get the sequence $A$, we must first get a $D$ sub-sequence, and then follow it up successfully (though not necessarily on the first attempt). Therefore expected number of rolls to get $A$

= number required to get $D$ + further number required to get from $D$ to $A$
= $k.DD$ + further numbers required to get from $D$ to $A$ (by the inductional hypothesis on theorem $X$ as $D$ has length $\leqslant n$).

Therefore the number of rolls to get from $B$ to $A$

= number to get from $D$ to $A$, as $D$ is the only useful bit of $B$
= number required to get $A$ from scratch $- k.DD$
= number required to get $A - k.BA$. (*iii*)

(*v*) The argument is essentially the same if $B$ has length $\leqslant n$. The only difference occurs if $BA$ has leading digit 1 making $D = B$; but it is still true that $D$ has length $\leqslant n$ allowing us to apply the inductional hypothesis.

(*vi*) If $A$ has length $\leqslant n$, $B$ must have length $n+1$

$\therefore$ $BA$ has leading digit 0,
$\therefore$ $D$ has length $\leqslant n$ and the earlier argument applies.

(*vii*) So far we have tacitly been assuming that $BA \neq 0$. If this is false, then there is no coincidence between any trailing sub-sequence of $B$ and a leading sub-sequence of $A$. Although we have $B$ to start with, it is of no assistance at all.

$\therefore$ number of rolls required to get from $B$ to $A$,
= number required to get $A$ from scratch
= number required to get $A - k.BA$ as $BA = 0$.

(*viii*) We now temporarily break off the induction argument on theorem $Y$ and apply it to theorem $X$. $A$ has length $n+1$, and consists of a single outcome $R$ followed by a sequence $A'$ of length $n$. By inspection $AA = k^n + A'A$. The probability of getting the sequence $A$ at any stage is $P(A) = 1/k^{n+1}$

$\therefore$ average distance between successive $A$ runs (as measured between their respective last terms) is $k^{n+1}$,
$\therefore$ $k^{n+1}$ = expected number of further rolls required to get from $A'$ to $A$
= number to get $A - kA'A$, (*v*)

$\therefore$ number of rolls required to get $A$ from scratch
$= k^{n+1} + k.A'A$
$= k(k^n + A'A) = k.AA$.

Thus the induction argument applies to theorem $X$.

(*ix*) Returning to theorem $Y$ we already know that the number of rolls required to get from $B$ to $A$
= number required to get $A - k.BA$.
But this now equals $k.AA - k.BA$, and so the induction applies to theorem $Y$ as well.

(*x*) Finally theorems $X$ and $Y$ are clearly true for $n = 1$; so they are true for all $n$.

**5.** We are now able to tackle the probabilities of the situation.

**Theorem Z**

The odds that the $B$ sequence precedes the $A$ sequence are given by $(AA - AB) : (BB - BA)$.

**Proof**

Suppose the whole experiment is carried out $N$ times, each time until both $A$ and $B$ appear; then if $P(B$ precedes $A) = q = 1 - p$, $B$ will precede $A$ in about $Nq$ of the occasions. The average waiting time to get $A$ is $k.AA$ and to get $B$ is $k.BB$

$\therefore$ average distance between occurrences
$= k(AA - BB)$, which can of course be positive or negative.

On those occasions when $B$ precedes $A$, the average distance is the waiting time to get from $B$ to $A$

$$= k.AA - k.BA.$$

When $A$ precedes $B$, the average distance

$= -$waiting time to get from $A$ to $B$
$= -(k.BB - k.AB)$.

The weighted average of these last two must give the over-all average;

$\therefore$ $k(AA - BB)$
$= q(k.AA - k.BA) - p(k.BB - k.AB)$.

$\therefore\ AA-BB = q.AA-q.BA-p.BB+p.AB,$

$\therefore\ p.AA-q.BB = p.AB-q.BA,$

$\therefore\ p(AA-AB) = q(BB-BA),$

$\therefore$ odds in favour of $B$ preceding $A$ are

$$q:p = (AA-AB):(BB-BA).$$

6. In theory, we can apply the above results to three person games. Thus suppose players $A, B, C$ all choose their sequences, and that the probabilities of the various finishing orders are given by

$$P(ABC) = p_1,\quad P(BCA) = p_3,\quad P(CAB) = p_5$$
$$P(ACB) = p_2,\quad P(BAC) = p_4,\quad P(CBA) = p_6.$$

Then immediately we have

$$p_1+p_2+p_3+p_4+p_5+p_6 = 1,$$
$$p_1+p_2+p_5 = P(A \text{ precedes } B),$$
$$p_3+p_4+p_6 = P(B \text{ precedes } A),$$
$$\therefore\ \frac{p_3+p_4+p_6}{p_1+p_2+p_5} = \text{odds on } B \text{ preceding } A$$
$$= \frac{AA-AB}{BB-BA}.$$

Similarly

$$\frac{p_2+p_5+p_6}{p_1+p_3+p_4} = \frac{BB-BC}{CC-CB}.$$

Again, in those games where $A$ wins, the waiting time before $B$ gets his sequence is $k(BB-AB)$ and before $C$ gets his is $k(CC-AC)$;

$\therefore$ average distance $= k(BB-AB-CC+AC)$, the positive direction being from $C$ to $B$.

But in these cases where $A$ wins, the conditional odds on $C$ beating $B$ are $p_2:p_1$;

$\therefore$ by the same argument as in theorem $Z$,

$$k(BB-AB-CC+AC)$$
$$= \frac{p_2 k}{p_1+p_2}(BB-CB) - \frac{p_1 k}{p_1+p_2}(CC-BC),$$
$$\therefore\ (p_1+p_2)(BB-AB-CC+AC) = p_2(BB-CB)-p_1(CC-BC),$$
$$\therefore\ p_1(BB-BC-AB+AC) = p_2(CC-CB-AC+AB).$$

With some licence, one could write this as

$$p_1(B-A)(B-C) = p_2(C-A)(C-B).$$

Similarly

$$p_3(C-B)(C-A) = p_4(A-B)(A-C),$$
$$p_5(A-C)(A-B) = p_6(B-C)(B-A).$$

The multiplication between the brackets is of course non-commutative. We now have six equations for the 6 unknown probabilities $p$.

7. Returning to two players only, let us examine the expected time to the compound event $A \cup B$. It would be $k.AA$ except for the fact that in a proportion $q$ of the time, $B$ precedes $A$ thereby decreasing the above value. The longer $A$ takes to occur on any particular occasion, the greater is the probability that $B$ will have preceded it; so the $A$ and $B$ events are dependent. Nevertheless in those cases where $B$ does occur first, the outcomes subsequent to the arrival of $B$ (and leading eventually to the occurrence of $A$) are completely random, and are independent of when the event $B$ occurred. The average time from $B$ to $A$ is $k(AA-BA)$; hence in the proportion $q$ of cases we overestimate the waiting time by the amount $k(AA-BA)$ if we persist with the value $k.AA$. Therefore the required answer is

$$k.AA-qk(AA-BA) = k(p.AA+q.BA):$$

equivalently it is $k(q.BB+p.AB)$. The fact that these two expressions are equal is equivalent to the statement of theorem $Z$.

8. By the same brand of argument, expected time to $A \cap B$ is

$$k.AA+pk(BB-AB)$$

or

$$k.BB+qk(AA-BA).$$

9. Another question is "what is the average time to get $A$ given that it precedes $B$." It is tempting to believe it is equal to the time to get $A \cup B$, and hence also to the average time to get $B$ given that it precedes $A$. This would be very tidy, but unfortunately it is not true. If

$$A \text{ is } HH,\quad k = 2,$$
$$B \text{ is } TH,\quad p = \tfrac{1}{4}.$$

The average time to get $A$ when it precedes $B$ is 2 because $A$ wins only when the first two outcomes are $HH$. The average time to get $B$ when it precedes $A$ is clearly greater than 2.

10. In a $k$-sided die situation, suppose that the two players can choose only between the sequences $A$ and $B$. It is supposed that the sequence chosen by the first player already exists "on the table," and that $Y$ pays to $X$ in pennies the waiting time from here to his own choice. Thus if

$$k = 3,$$
$$A \text{ is } 213$$
$$B \text{ is } 121$$

then

| | |
|---|---|
| $100 = 9$ | $000 = 0$ |
| $A-213$ | $A-213$ |
| $A-213$ | $B-121$ |
| $101 = 10$ | $010 = 3$ |
| $B-121$ | $B-121$ |
| $B-121$ | $A-213$ |

and expectation pay-off matrix (from columns player to rows player) is

column choice

$$\text{row choice}\quad \begin{array}{c} \\ A \\ B \end{array}\begin{array}{cc} A & B \\ \left[\begin{array}{c} k(AA-AA) \\ k(AA-BA) \end{array}\right. & \left.\begin{array}{c} k(BB-AB) \\ k(BB-BB) \end{array}\right] \end{array} = \begin{bmatrix} 0 & 30 \\ 18 & 0 \end{bmatrix}.$$

Under standard mixed strategy game theory, the rows player should choose at random between first and second in the ratio $3:5$.

Another way of analysing the game is for the rows player to argue:

"If column $A$ is selected, I ought to get $k\,.\,AA$. The reason I do not is that my pre-sequence gives assistance which I grudge. If I choose $A$, my pay-off is $k(AA-AB)$ so my assistance amounts to $k\,.\,AA$. On the other hand with row $B$ my assistance is only $k\,.\,BA$. I am hanged if I am going to assist my competitor any more than necessary. My strategy must therefore be to minimise from the pay-off matrix

$$\begin{bmatrix} AA & AB \\ BA & BB \end{bmatrix}.$$

This is achieved by a random selection of rows with odds $(BB-BA):(AA-AB)$."

I think that such a player, one who does down his opponent even at the cost of his own personal gain, could be called spiteful.

Now let us change the scenario to where the row is selected not by human intellect but by blind chance. The obvious way of setting this up is to run a random sequence of rolls and to choose $A$ when its precedes $B$, and $B$ when it precedes $A$. By theorem $Z$ this results in the odds $A:B=(BB-BA):(AA-AB)$. But these are precisely the odds to minimise rows assistance. Does this prove that blind chance is spiteful? Many of us have had occasions in our lives to suspect as much. Now we know!

Returning to our numerical example,

$$\begin{bmatrix} AA & AB \\ BA & BB \end{bmatrix} = \begin{bmatrix} 9 & 0 \\ 3 & 10 \end{bmatrix}$$

and the chance odds are

$$A:B = 7:9.$$

**11.** We have proved for a two person wager that the odds for precedence are the same as the spiteful odds in a mixed strategy game. Perhaps the equivalent argument can also go the other way. Let us test the situation in a simple numerical case with three players.

Take coin tossing where

player $A$ chooses $HH$,
player $B$ chooses $HT$,
player $C$ chooses $TT$.

The equivalent pay-off matrix is

$$\begin{bmatrix} AA & AB & AC \\ BA & BB & BC \\ CA & CB & CC \end{bmatrix} = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix}.$$

Odds must be selected for the rows to give the same return whichever column is chosen. Therefore

$$3p = p+2q = q+3r,$$

$$\therefore\ p=q, \quad 2p=3r,$$

$$\therefore\ p:q:r = 3:3:2.$$

It is easily seen in this case that any variation in these odds allows "columns" to pay out less on average.

In (6), we had equations for the probabilities of the various possible results with players $A, B, C$. Filling numerical values into those equations gives

$$p_1+p_2+p_3+p_4+p_5+p_6 = 1,$$

$$\frac{p_3+p_4+p_6}{p_1+p_2+p_5} = \frac{AA-AB}{BB-BA} = \frac{2}{2} = 1,$$

$$\frac{p_2+p_5+p_6}{p_1+p_3+p_4} = \frac{1}{3},$$

$$p_1(2-1-1+0) = p_2(3-0-0+1),$$
$$p_3(3-0-1+0) = p_4(3-0-0+1),$$
$$p_5(3-1-0+0) = p_6(2-0-0+0).$$

Simplifying,

$$p_1+p_2+p_3+p_4+p_5+p_6 = 1,$$
$$p_3+p_4+p_6 = p_1+p_2+p_5 = \tfrac{1}{2},$$
$$p_1+p_3+p_4 = 3p_2+3p_5+3p_6,$$
$$p_2 = 0,$$
$$p_3 = 2p_4,$$
$$p_5 = p_6,$$

$$\therefore\ p_1 = \tfrac{3}{8}, \quad p_3 = \tfrac{2}{8}, \quad p_5 = \tfrac{1}{8},$$
$$p_2 = 0, \quad p_4 = \tfrac{1}{8}, \quad p_6 = \tfrac{1}{8},$$

$$P(A \text{ wins}) = p_1+p_2 = \tfrac{3}{8},$$
$$P(B \text{ wins}) = p_3+p_4 = \tfrac{3}{8},$$
$$P(C \text{ wins}) = p_5+p_6 = \tfrac{2}{8}.$$

giving exactly the same odds as above.

**12.** In (2) we considered each possible sub-sequence of length 3, located its best beater, and drew a diagram (Fig. 1) which contained an intransitive ring. Doing the same for length 4 gives (Fig. 2):

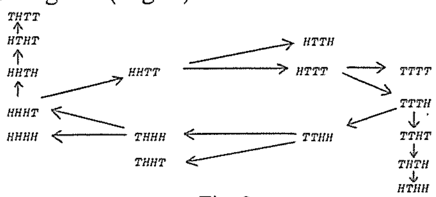


Fig. 2

Length 5 gives something even more complicated.

**13.** The above figure contains an intransitive hexagon. If we drop the condition about noting only *best* beaters, we can have intransitive polygons of other sizes, the largest is the 14-gon.

$$HHHT \to THHT \to HHTT \to TTHT \to HTHT \to$$
$$THTT \to HTTT \to TTTH \to HTTH \to TTHH \to$$
$$HHTH \to THTH \to HTHH \to THHH (\to HHHT).$$

The smallest are triangles; in fact we can get these from alternate vertices of the above hexagon

$$HHHT \to HTTT \to TTHH (\to HHHT)$$
$$TTTH \to THHH \to HHTT (\to TTTH).$$

**14.** It can be shown that the best choices for $A$ against a purely random choice by $B$ are $HTTT$ and $THHH$. Next come the pair $HHTT$ and $TTHH$, and then the pair $HHHT$ and $TTTH$. It is interesting that these six are the vertices of the above intransitive hexagon. But while this list gives the preference order

$HTTT$
$HHTT$
$HHHT$

it is apparent from the intransitive hexagon that on average $HHHT$ beats $HHTT$ and $HHTT$ beats $HTTT$ (which is the exactly reversed order!).

**15.** We turn now to the recent *Bulletin* article[2] on "Beetle." We shall adopt the same notation except that $W_{a,b}$ for us will denote the expected *total* number of rolls of an ordinary die to obtain $a$ 5's and $b$ 6's in any order (and not necessarily consecutively). In other words we proceed from $0, 0$ to $a, b$; also we work in terms of single rolls, not blocks of 6.

**16.** To get the compound event $5 \cup 6$, we expect to need 3 rolls. Therefore to get $2n$ of these results, we need $6n$ rolls. But however many rolls we in fact take, we shall be at the point

$$\begin{array}{lll} (0, 2n) & \text{with probability} & \binom{2n}{0}\Big/2^{2n} \\ (1, 2n-1) & \text{with probability} & \binom{2n}{1}\Big/2^{2n} \\ (2, 2n-2) & \text{with probability} & \binom{2n}{2}\Big/2^{2n} \\ \vdots & & \\ (n, n) & \text{with probability} & \binom{2n}{n}\Big/2^{2n} \\ \vdots & & \\ (2n, 0) & \text{with probability} & \binom{2n}{2n}\Big/2^{2n} \end{array}$$

$$\begin{aligned} \therefore\ W_{n,n} &= 6n + 6n\binom{2n}{0}\Big/4^n \\ &\quad + 6(n-1)\binom{2n}{1}\Big/4^n + \ldots + 6\binom{2n}{n-1}\Big/4^n \\ &\quad + 0\,.\binom{2n}{n}\Big/4^n + 6\binom{2n}{n+1}\Big/4^n \\ &\quad + \ldots + 6n\binom{2n}{2n}\Big/4^n \end{aligned}$$

because from

$(0, 2n)$ we still need $n$ 5's

$(1, 2n-1)$ we still need $(n-1)$ 5's

$\vdots$

$(n, n)$ we need nothing more

$\vdots$

$(2n, 0)$ we still need $n$ 6's,

$$\begin{aligned} \therefore\ W_{n,n} &= 6n + \frac{2\,.\,6}{4^n}\left\{n\binom{2n}{0} + (n-1)\binom{2n}{1} \right. \\ &\quad \left. + \ldots + 1\binom{2n}{n-1}\right\} \\ &= 6n + \frac{12}{4^n}K_n, \end{aligned}$$

where

$$\begin{aligned} K_n &= n\left\{\binom{2n}{0} + \binom{2n}{1} + \ldots + \binom{2n}{n}\right\} \\ &\quad - \left\{0\,.\binom{2n}{0} + 1\binom{2n}{1} + \ldots + n\binom{2n}{n}\right\} \\ &= n\left\{\tfrac{1}{2}\,.\,2^{2n} + \frac{1}{2}\binom{2n}{n}\right\} \\ &\quad - \left\{\frac{(2n)!}{0!\,(2n-1)!} + \frac{(2n)!}{1!\,(2n-2)!} + \ldots + \frac{(2n)!}{(n-1)!\,n!}\right\} \\ &= \frac{n}{2}\left\{4^n + \binom{2n}{n}\right\} - 2n\left\{\binom{2n-1}{0} \right. \\ &\quad \left. + \binom{2n-1}{1} + \ldots + \binom{2n-1}{n-1}\right\} \\ &= \frac{n}{2}\left\{4^n + \binom{2n}{n}\right\} - 2n\{\tfrac{1}{2}\,.\,2^{2n-1}\} \end{aligned}$$

$$\begin{aligned} \therefore\ W_{n,n} &= 6n + \frac{12}{4^n}\,.\,\frac{n}{2}\left\{4^n + \binom{2n}{n}\right\} - \frac{12}{4^n}\,.\,n\,.\,2^{2n-1} \\ &= 6n + 6n + 6n\binom{2n}{n}\Big/4^n - 6n \\ &= 6n + 6n\binom{2n}{n}\Big/4^n. \end{aligned}$$

**17.**

$$\begin{aligned} W_{n,n} &= 3 + \tfrac{1}{2}W_{n-1,n} + \tfrac{1}{2}W_{n,n-1} \\ &= 3 + W_{n-1,n} \quad \text{by symmetry,} \\ \therefore\ W_{n-1,n} &= W_{n,n} - 3. \end{aligned}$$

**18.** We know that

$$\begin{aligned} W_{n,1} &= 6n + 6/2^n, \\ W_{n,2} &= 6n + 6(n+4)/2^{n+1}. \end{aligned}$$

Suppose inductionally that

$$W_{n,3} = 6n + 3(n^2 + 9n + 24)/2^{n+2}.$$

This is true for $n = 1$, as then

$$\begin{aligned} W_{1,3} &= 6 + 3(1 + 9 + 24)/2^3 \\ &= 6 + \frac{3\,.\,34}{8} = 18\tfrac{3}{4} = W_{3,1}. \end{aligned}$$

Also,

$$\begin{aligned} W_{n+1,3} &= 3 + \tfrac{1}{2}W_{n,3} + \tfrac{1}{2}W_{n+1,2} \\ &= 3 + 3n + 3(n^2 + 9n + 24)/2^{n+3} \\ &\quad + 3(n+1) + 6(n+5)/2^{n+3} \\ &= 6(n+1) + 3\{n^2 + 9n + 24 + 2(n+5)\}/2^{n+3} \\ &= 6(n+1) + 3\{(n+1)^2 + 9(n+1) + 24\}/2^{n+3} \end{aligned}$$

which completes the induction argument.

Again, suppose inductionally that

$$W_{n,4} = 6n + (n^3 + 15n^2 + 86n + 192)/2^{n+3}.$$

This is true for $n = 1$, as then

$$W_{1,4} = 6 + 294/2^4 = 24\tfrac{3}{8} = W_{4,1}.$$

Also

$$\begin{aligned} W_{n+1,4} &= 3 + \tfrac{1}{2}W_{n,4} + \tfrac{1}{2}W_{n+1,3} \\ &= 3 + 3n + (n^3 + 15n^2 + 86n + 192)/2^{n+4} + 3(n+1) \end{aligned}$$

$$+3(n^2+11n+34)/2^{n+4}$$
$$=6(n+1)+(n^3+18n^2+119n+294)/2^{n+4}$$
$$=6(n+1)+\{(n+1)^3+15(n+1)^2+86(n+1)+192\}/2^{n+4}$$

which completes the induction argument. It is obvious from this that $W_{n,r}$ contains a polynomial in $n$ of degree $r-1$, but it is not obvious what that polynomial is.

**19.** In 4 (*iii*) of the paper, the stated waiting time of $n^2+1$ for 3 consecutive wins out of $n$ equally competent teams is surely a misprint for $n^2+n+1$.

**20.** From the same sub-section, a closer upper bound for the waiting time for *any* 3 wins is $1+2n$. For $n = 2$ this gives 5, whereas the exact answer is $4\frac{1}{8}$. For $n = 3$, the answer is 5·05 (E. and O.E.).


**References**

1. Gardner, M., *Scientific American*, October 1974, pp. 120–125.
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**THE INSTITUTE OF MATHEMATICS AND ITS APPLICATIONS**

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