# The Weirdness of Number 6174

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Abstract

The number 6174 is a really mysterious number. At first glance, it might not seem so obvious, but as we are about to see, anyone who can subtract can uncover the mystery that makes 6174 so special.

Keywords: 6174, Kaprekar operation, Kaprekar constant, Number theory

## 1. The Kaprekar operation

6174 is truly a strange number. It is also a number with which we share a close relationship from elementary school up to university. But before explaining what kind of number it is, would you mind doing a little simple arithmetic?

First choose a single 4 digit number. When choosing, please avoid numbers with four identical digits like 1111 or 2222. For example, let's consider the current year, 2005. Take the four digits that compose the number and reorder them into the largest and smallest numbers possible. For numbers with less than 4 digits pad the left hand side with zeroes to maintain 4 digits. In the case of 2005, the results are 5200 and 0025. Taking the difference between the largest and the smallest yields

$5200-0025=5175.$

This type of operation is known as a Kaprekar operation. The name derives from the name is an Indian mathematician, D.R. Kaprekar, who discovered a special property of the number 6174. Iterating the operation on our newly revealed number yields,

$7551-1557=5994$
$9954-4599=5355$
$5553-3555=1998$
$9981-1899=8082$
$8820-0288=8532$
$8532-2358=6174$
$7641-1467=6174.$

When the number becomes 6174, the operation repeats and 6174 thus cycles, and is known as the 'kernel'. No matter what the initial number may be, the sequence will eventually arrive at 6174. In fact, the kernel number 6174 will definitely be reached. If you remain doubtful, try the process again with a different number. 1789 develops as follows.

$9871-1789=8082$
$8820-0288=8532$
$8532-2358=6174$

2005 reaches 6174 after the Kaprekar operation is applied 7 times. 1789 reaches it after 3 times. This works for all 4 digit numbers. Isn't this strange? For elementary school pupils this is good practice for subtracting 4 digit numbers. For university students, thinking about why this happens reveals that 6174 is an exceptionally fascinating number. From this point on, I'd like to take a close look at the background of this number.

## 2. Solution using simultaneous linear equations

Let the largest number formed by rearranging the 4 digits be represented as *abcd*, and the smallest as *dcba*. Since the solution is cyclic, the difference between these two may be expressed as a combination of the digits $\{a, b, c, d\}$.

With $9 \geq a \geq b \geq c \geq d \geq 0$ and the subtraction

$$\begin{array}{rcccc} & a & b & c & d \\ - & d & c & b & a \\ \hline & A & B & C & D \end{array}$$

the differences between each digit obey the following relationships.

$$D = 10 + d - a \qquad (a > d)$$
$$C = 10 + c - 1 - b = 9 + c - b \qquad (b > c - 1)$$
$$B = b - 1 - c \qquad (b > c)$$
$$A = a - d.$$

Consider the relationship between the value of $(A, B, C, D)$ and the set $\{a, b, c, d\}$. Since there are 4 equations and 4 variables this is a 4 dimensional simultaneous linear equation. It ought to have a solution. Calculating the number of permutations of the elements of $\{a, b, c, d\}$ yields $4! = 24$ alternatives. It is sufficient to test each of these. The details are omitted but the unique solution of this simultaneous linear equation occurs when $(A, B, C, D) = (b, d, a, c)$. Solving this we obtain

$$(a, b, c, d) = (7, 6, 4, 1).$$

$abcd - dcba = bdac$, *i.e.*, $7641 - 1467 = 6174$, and the kernel number is 6174.

This phenomenon occurring with 4 digit numbers is also known to occur with 3 digit numbers. For example, with the 3 digit number 753, the calculation is as follows.

$$753 - 357 = 396$$
$$963 - 369 = 594$$
$$954 - 459 = 495$$
$$954 - 459 = 495$$

In the 3 digit case, the number 495 is reached, and this occurs for all 3 digit numbers. Why don't you try some?

## 3. The number of iterations needed to reach 6174

I first heard about the number 6174 from an acquaintance in around 1975 and it made a strong impression on me. I was surprised by the beautiful fact that all 4 digit numbers reach 6174, and thought it might be possible to prove this easily using high school level mathematical knowledge. But the calculations are surprisingly complex and I left the problem in an unsolved

| Number of iterations | Frequency |
|---:|---:|
| 0 | 1 |
| 1 | 356 |
| 2 | 519 |
| 3 | 2124 |
| 4 | 1124 |
| 5 | 1379 |
| 6 | 1508 |
| 7 | 1980 |
| Total | 8991 |

Table 1. Number of iterations needed to reach 6174

state. At that time I made a copy of a journal paper about the topic. I attempted to investigate the upper limit on the number of iterations necessary to settle on the number 6174 using a computer. By means of a Visual Basic program with about 50 lines, I tested all the 4 digit numbers between 1000 and 9999. This included all 8991 digit natural numbers excluding those with 4 equal digits (1111, 2222, ..., 9999). Table 1 shows the frequency of each number of iterations needed to reach 6174. The largest number of steps needed is 7. If 6174 is not reached in 7 iterations, then a mistake was made during calculation. This is a useful educational material for elementary school students to practice subtracting 4 digit numbers. For the initial number 6174, without even performing any Kaprekar operations, 6174 has already been reached so in this case the number of iterations is taken as 0.

## 4. The route to 6174

Regarding D.R. Kaprekar, details may be found in Lines, M. (see Lines, 1986). Kaprekar was an Indian mathematician active in the 1940s. The aspects of the problem are explained as follows.

Taking an arbitrary 4 digit number expressed as $abcd$ (where $a \geq b \geq c \geq d$) and executing the first subtraction may be considered as follows. The largest 4 digit number is equal to $1000a+100b+10c+d$, so the smallest number is $1000d+100c+10b+a$. Subtracting the smallest number from the largest and combining similar terms yields the following.

$$\begin{aligned} &1000a+100b+10c+d-(1000d+100c+10b+a) \\ &=1000(a-d)+100(b-c)+10(c-b)+(d-a) \\ &=999(a-d)+90(b-c) \end{aligned}$$

$a-d$ has a value between 1 and 9. $b-c$ takes an arbitrary value between 0 and 9, so in total there are 90 numbers taking the form above. Table 2 was produced in order to confirm this fact.

In this table , so the 36 entries in the bottom left (a catch-all case) are meaningless numbers. Next we will execute the second subtraction, so the numbers in Table 2 are rearranged into the corresponding largest values as shown in Table 3.

Ignoring the repetitions of the catch-all cases, there are 30 entries remaining in this table.

| | $999\times(a-d)$ 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 999 | 1998 | 2997 | 3996 | 4995 | 5994 | 6993 | 7992 | 8991 |
| 1 | 1089 | 2088 | 3087 | 4086 | 5085 | 6084 | 7083 | 8082 | 9081 |
| 2 | 1179 | 2178 | 3177 | 4176 | 5175 | 6174 | 7173 | 8172 | 9171 |
| 3 | 1269 | 2268 | 3267 | 4266 | 5265 | 6264 | 7263 | 8262 | 9261 |
| $90\times$ $(b-c)$ 4 | 1359 | 2358 | 3357 | 4356 | 5355 | 6354 | 7353 | 8352 | 9351 |
| 5 | 1449 | 2448 | 3447 | 4446 | 5445 | 6444 | 7443 | 8442 | 9441 |
| 6 | 1539 | 2538 | 3537 | 4536 | 5535 | 6534 | 7533 | 8532 | 9531 |
| 7 | 1629 | 2628 | 3627 | 4626 | 5625 | 6624 | 7623 | 8622 | 9621 |
| 8 | 1719 | 2718 | 3717 | 4716 | 5715 | 6714 | 7713 | 8712 | 9711 |
| 9 | 1809 | 2808 | 3807 | 4806 | 5805 | 6804 | 7803 | 8802 | 9801 |

Table 2. The numbers after the first subtraction

| | $999\times(a-d)$ 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 9990 | 9981 | 9972 | 9963 | 9954 | 9954 | 9963 | 9972 | 9981 |
| 1 | 9810 | 8820 | 8730 | 8640 | 8550 | 8640 | 8730 | 8820 | 9810 |
| 2 | | 8721 | 7731 | 7641 | 7551 | 7641 | 7731 | 8721 | 9711 |
| 3 | | | 7632 | 6642 | 6552 | 6642 | 7632 | 8622 | 9621 |
| $90\times$ $(b-c)$ 4 | | | | 6543 | 5553 | 6543 | 7533 | 8532 | 9531 |
| 5 | | | | | 5544 | 6444 | 7443 | 8442 | 9441 |
| 6 | | | | | | 6543 | 7533 | 8532 | 9531 |
| 7 | | | | | | | 7632 | 8622 | 9621 |
| 8 | | | | | | | | 8712 | 9711 |
| 9 | | | | | | | | | 9801 |

Table 3. The numbers prior to the second subtraction

Figure 1 shows a schematic diagram of the ways in which these 30 numbers reach 6174. According to this diagram it should be possible to understand at a glance that all 4 digit natural numbers reach 6174. It can also be seen that at most 7 iterations are necessary. Even so, it's certainly strange. Was Kaprekar, who discovered this, a man of exceptional intelligence or was he a man of exceptional leisure?

## 5. Recurring decimals

Real numbers include both rational and irrational numbers. Rational numbers are those that can be expressed as a fraction $\frac{n}{m}$ ($m$, $n$ are integers and $m\neq 0$), while irrational numbers are those that cannot be expressed in this way.

Real numbers may be written as decimals. Rational numbers are finite or recurring decimals.

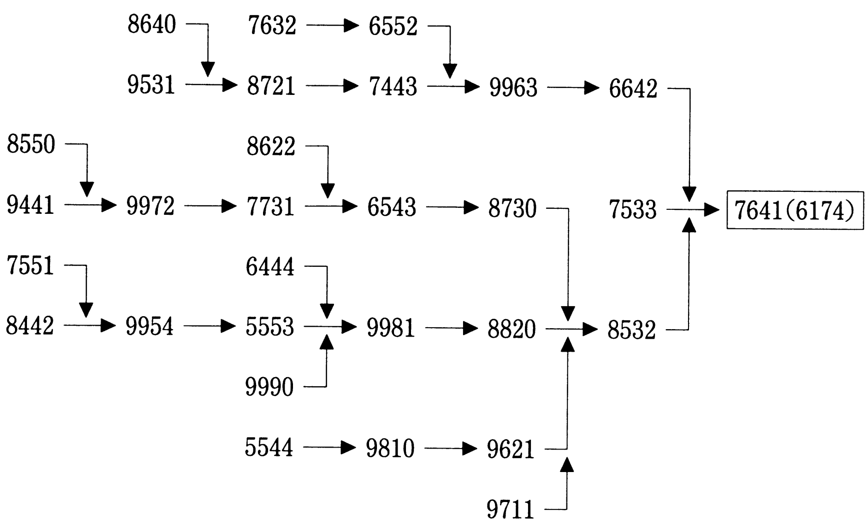


Figure 1. Schematic diagram leading to 7641 (6174)

Irrational numbers on the other hand, are non-cycling infinite decimals. For example, the following numbers are irrational.

$\sqrt{2}=1.41421356237309504880168872420\cdots$

$\pi=3.14159265358979323846264338327\cdots$

Recurring decimals, which are rational numbers, are explained as follows. Recurring decimals are those infinite decimals for which after some point a sequence of digits (the recurring sequence) repeats indefinitely. When writing recurring decimals, a dot is marked above each end of the recurring sequence when it first appears. The remaining digits are omitted. For example,

$0.7\dot{2}1\dot{4}=0.7214214214\cdots$

is such a recurring decimal. By expressing this number using a geometrical progression, and using the formula for geometrical progressions, it is possible to find a corresponding fraction.

$$
\begin{aligned}
0.7\dot{2}1\dot{4} &= \frac{7}{10}+\frac{214}{10^4}+\frac{214}{10^7}+\cdots \\
&= \frac{7}{10}+\frac{214}{10^4}\left(1+\frac{1}{10^3}+\frac{1}{10^6}+\cdots\right) \\
&= \frac{7}{10}+\frac{214}{10^4}\times\frac{1}{1-\frac{1}{10^3}} \\
&= \frac{7}{10}+\frac{214}{10(10^3-1)}=\frac{7207}{9990}
\end{aligned}
$$

The denominator is $9990=2\times3^3\times5\times37$.

This recurring decimal is a rational number whose denominator contains other prime factors besides 2 and 5. Rational numbers may be classified as follows by examining the prime factorization of the denominator.

Finite decimals: factors include 2 and 5 alone
Pure recurring decimals: factors do not include 2 or 5
Mixed recurring decimals: factors include 2 or 5 as well as other factors

Pure recurring decimals are formed from the recurring sequence alone, while mixed recurring decimals also include another part besides the recurring sequence. For example, $\frac{1}{4}=0.25$ is a finite decimal, $\frac{1}{7}=0.\dot{1}4285\dot{7}$ is a pure recurring decimal, and $\frac{1}{12}=0.08\dot{3}$ is a mixed recurring decimal, because $4=2^2$, $7=7$ and $12=2^2\times3$.

The Kaprekar operation may be applied to these recurring decimals. When the number has 3 or 4 digits, after a finite number of iterations the numbers 495 and 6174 are reached respectively, so the sequence has a form like a single mixed recurring decimal.

## 6. What happens with 2 digits and with 5 or more digits?

Numbers with 4 or 3 digits converge on a unique number, but what happens in the case of 2 digits? For example, starting with 28 and repeating the largest-minus-smallest Kaprekar operation yields

| | | |
|---|---|---|
| 28 | $82-28=54$ | $54-45=9$ |
| $90-09=81$ | $81-18=63$ | $63-36=27$ |
| $72-27=45$ | $54-45=9$, | |

which beginning with 9, cycles in the pattern $9 \rightarrow 81 \rightarrow 63 \rightarrow 27 \rightarrow 45 \rightarrow 9$. Thus for numbers of 2 digits, a certain domain cycles in a similar fashion to two mixed recurring decimals.

Next, what happens with 5 digit numbers? Firstly, isn't there a kernel value like 6174 and 495? Expressing a five digit number using $9\geq a\geq b\geq c\geq d\geq e\geq 0$, the largest minus the smallest may be written as $abcde-edcba=ABCDE$, where $(A, B, C, D, E)$ is chosen from the 120 permutations of $\{a, b, c, d, e\}$. This is a constrained case-based simultaneous linear equation. Regarding the 5 digit Kaprekar problem, a considerable amount of computation has already been performed and as a consequence it is known that there is no kernel value, and all 5 digit numbers enter one of the following loops.

$71973 \rightarrow 83952 \rightarrow 74943 \rightarrow 62964$
$75933 \rightarrow 63954 \rightarrow 61974 \rightarrow 82962$
$59994 \rightarrow 53955$

Regarding integers with 6 or more digits, Malcolm Lines indicates that increasing the number of digits soon becomes a tedious issue merely increasing the computation time. The existence of kernel values is summarized in Table 4.

This table reveals that for 6 and 8 digits, there are 2 kernel values, and in some cases one of the kernel values is reached while in others cases the sequence enters a loop. For a computer with 32 bit words, integers are represented using 32 bits, so it is possible perform calculations up to $2^{31}-1(=2147483647)$ which is around the beginning of 10 digit numbers. Just as described by Malcolm Lines, it began to seem nonsensical, so I stopped calculating.

I wanted to know more about the roots of this problem, and investigated a little further. I

| digits | Kernel values | |
|---|---|---|
| 2 | Nothing | |
| 3 | 495 | Unique |
| 4 | 6174 | Unique |
| 5 | Nothing | |
| 6 | 549945, 631764 | |
| 7 | Nothing | |
| 8 | 63317664, 97508421 | |

Table 4. Kernel values

encountered Martin Gardner's book by chance, and came to understand the situation in this area (see Gardner, 1978). The explanation at the end of the book states that the number 6174 is called Kaprekar's constant after an Indian named Dattathreya Ramachandra Kaprekar, that he was the first person to demonstrate its importance ("Another Solitaire Game", Scripta Mathematica, 15 (1949), pp 244-245), and that he subsequently presented "An Interesting Property of the Number 6174" (1955), "The New Constant 6174" (1959), and "The Mathematics of the New Self Numbers" (1963). It seems that these represent the truth of the matter, and computer-based revelations have placed the number under a spotlight of attention.

Martin Gardner takes it that for 1, 2, 5, 6 and 7 digit numbers, Kaprekar's constant does not exist. However, as shown above, for 6 digit numbers there are two kernel values (these cases are not known as Kaprekar's constants). Martin Gardner also states kernel values for 8, 9 and 10 digit numbers. For 8 digits it is 97508421, for 9 digits, 864197532, and for 10 digits, 9753086421. The subtractions are as follows.

$$98754210 - 01245789 = 97508421$$
$$987654321 - 123456789 = 864197532$$
$$9876543210 - 0123456789 = 9753086421$$

The form is beautiful in the 9 and 10 digit cases, and these were probably obtained intuitively. Perhaps the value for the 8 digit case was obtained through computer processing. It might alternatively have been found using a calculator or pencil and paper arithmetic. It is sufficient if there is a way to find out without resorting to computer power. If I have another opportunity I'll try investigating using a different method. This problem remains captivating, and I'll mention David Wells' book, which is of historical interest and discusses each of the numbers (see Wells, 1987).

## 7. Chance or necessity?

The existence of cyclic numbers is made clear by the simultaneous linear equations. It was proven that for 3 digit numbers 495 is a unique cyclic number and likewise 6174 for 4 digit numbers. It was also confirmed that all 3 digit numbers converge on 495 and that all 4 digit numbers converge on 6174. However, this was merely demonstrated, and I think that the real reason why all the numbers converge on the cyclic number has not been demonstrated.

Is it by chance or by necessity that it only works for 3 and 4 digit numbers? I have a feeling that it is a matter of chance. Allow me to introduce the following puzzle which has already been solved (see Nishiyama, 1989).

$$\begin{array}{r} \square\square\square\square\square \\ \times \quad \square\square\square\square\square \\ \hline 1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9 \end{array}$$

This worm-eaten arithmetic puzzle involves putting numbers in the blank boxes, but the form is so beautiful that I had hoped in my heart that perhaps some great theory of numbers lay hidden within, but I found out that it is merely coincidental. It has been confirmed that there is a host of such puzzles in existence.

$$\begin{array}{r} \square\square\square\square\square \\ \times \quad \square\square\square\square\square \\ \hline 1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 4 \end{array}$$

Kaprekar's problem may be thought of as a similar type to this worm-eaten arithmetic problem. The trick for solving this problem is to use the prime factorization. Applying this to the first example,

$$123456789 = 3 \times 3 \times 3607 \times 3803,$$

and the answer is $10821 \times 11409$. Applying the same method to the latter,

$$123456784 = 2^4 \times 11^2 \times 43 \times 1483,$$

and the answer is $10406 \times 11864$.

Historically, some of the developments in science and mathematics have been prompted by 'mistakes'. For the worm-eaten arithmetic problem, given the former case the desire to try and solve it occurs naturally, while in the latter case one probably wouldn't be particularly interested. The reason is that the former statement appears so beautiful. Just knowing that under Kaprekar operations all 4 digit numbers converge on 6174 and all 3 digit numbers on 495 provides sufficient charm as a mathematical puzzle. Who could say that this is merely a coincidence? I hope that some great theory of mathematics might lie behind it. This hope might end up being a beautiful mistake, but that's something I don't wish to believe.

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