Humble-Nishiyama Randomness Game - A New Variation on Penney's Coin Game

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This paper offers a variation on Penney's Game using a pack of ordinary playing cards. The ***Humble-Nishiyama Randomness Game*** *follows the same format using Red and Black cards, instead of Heads and Tails. The card game's finite structure creates a much greater chance of the 2nd player winning.*

Is randomness merely the human inability to recognise a pattern that may in fact exist?

Understanding if events are random or have some underlying structure is a fascinating area of mathematics, filled with great discoveries.

The Philosopher Francis Bacon said "The human understanding, once it has adopted an opinion, collects any instances that confirm it, and though the contrary instances may be more numerous, either does not notice them or else rejects them, in order that this opinion will remain unshaken" [1].

A game that was created by Walter Penney in 1969 [2], is based on observing the occurrence of groups of heads and tails when repeatedly throwing a coin. Let your opponent (1st player) select any sequence of three coins and then, referring to the table below, you choose the relevant 2nd players choice next to it according to the chart. You then record a sequence of coin throws looking for one of your three coin sequences in the long chain of throws, such as HTHTHHHHTHHHHTTTTHTHH. The winner is the person whose pattern appears first.

At first glance you would think that the game is completely fair and not biased in any way, but in fact whatever sequence is selected by your opponent, you can always select a sequence which is more likely to appear first.

The maths of this order from apparent randomness can be seen by looking at the following three cases:

- If your opponent chooses HHH, you then choose THH (as in the table). The one chance in eight that the first three tosses of the coin is HHH, your opponent wins straight away. Yet in all other cases, if HHH is not in the first three tosses of the coin, then THH will occur first.
- If your opponent chooses HHT, you then choose THH. The chance that HHT occurs first is conditional on either getting HHT or HHHT or HHHHT etc:

$$\text{P(HHT before THH)} = \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \ldots = \frac{1/8}{(1-1/2)} = \frac{1}{4}.$$

Hence $\text{P(THH before HHT)} = 1 - \frac{1}{4} = \frac{3}{4}.$

- If your opponent chooses HTH, you then choose HHT.
  Let x = P(HHT comes before HTH).
  Ignore any leading T's. If we look to see what happens after the first H, on average half the time the next throw is H, and then HHT is more likely to occur before HTH.
  Half the time on average the next throw is T, but if this is followed by another T, we are back to the beginning, hence you can write

$$x = \frac{1}{2} + \frac{1}{2}\cdot\frac{1}{2}\cdot x.$$

This gives, $\frac{3}{4}x = \frac{1}{2}$, so $x = \frac{2}{3}$.

1st player's choice	2nd player's choice	Odds in favour of 2nd player
HHH	THH	7 to 1
HHT	THH	3 to 1
HTH	HHT	2 to 1
HTT	HHT	2 to 1
THH	TTH	2 to 1
THT	TTH	2 to 1
TTH	HTT	3 to 1
TTT	HTT	7 to 1

You can see from the table that your choice as the second player has a greater chance of appearing before your opponents in each case. This is why on average the second player should win over a group of say ten games. As well as looking at the theory, students should be encouraged to play the game. This practical aspect of mathematical development is often overlooked in education and can lead to a deeper understanding of the subject [3].

Penney's surprising result generated a succession of papers in the seventies and eighties [4 - 11].

The authors suggest a variation on Penney's Game using a pack of ordinary playing cards. The **Humble-Nishiyama Randomness Game** follows the same format using Red and Black cards, instead of Heads and Tails. At the start of a game each player decides on their three colour sequence for the whole game. Every time the 1st or 2nd player's sequence of cards appears, all those cards are removed from the game as a "winning trick". This continues until the full pack of 52 cards is used. At the end the player with the most tricks is declared the winner. An average game will consist of around 7 tricks. Due to the finite number of cards in a pack you can show that the second player's chance of winning is greatly increased.

Below are the results from a computer simulation of 1000 games, each game is played with a full pack of cards:

BBB vs RBB - RBB wins 995 times, 4 draws, BBB wins once
BBR vs RBB - RBB wins 930 times, 40 draws, BBR wins 30 times
BRB vs BBR - BBR wins 805 times, 79 draws, RBR wins 116 times
RBB vs RRB - RRB wins 890 times, 68 draws, RBB wins 42 times
BRR vs BBR - BBR wins 872 times, 65 draws, BRR wins 63 times
RBR vs RRB - RRB wins 792 times, 85 draws, RBR wins 123 times
RRB vs BRR - BRR wins 922 times, 51 draws, RRB wins 27 times
RRR vs BRR - BRR wins 988 times, 6 draws, RRR wins 6 times

To obtain an understanding of why the card game's finite structure creates a much greater chance of the 2nd player winning, you can use probability theory. Cases such as RRR vs BRR show a higher probability of BRR occurring than HTT in the game TTT vs HTT. Yet for most games the probability calculations become too difficult to manage.

The idea that the 2nd player's chance of winning with cards is greater than with coins can be seen by looking at the recursive technique which was used to explain the coin game mentioned previously.

Looking at certain cases you can form equations such as these:

Assuming that there is a larger quantity of Black cards left in the pack and looking at RBR vs RRB gives:

X_B = (Prob less than 0.5) + (Prob greater 0.5)(Prob greater than 0.5) X_B.

A higher quantity of Red cards left in the pack gives:

X_R = (Prob greater than 0.5) + (Prob less than 0.5)(Prob less than 0.5) X_R.

Another way to validate the computer simulation is to assume for simplicity in the game RBR vs RRB, that the probability of RBR is at most 1/3 and RRB is at least 2/3.

Let P(RRB,*n*), the probability that RRB wins the most tricks after *n* matches in a game, be

$$P(RRB, 1) = 2/3,$$

$$P(RRB, 3) = {}^3C_3\left(\frac{2}{3}\right)^3 + {}^3C_2\left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right) = 0.741,$$

$$P(RRB, 5) = {}^5C_5\left(\frac{2}{3}\right)^5 + {}^5C_4\left(\frac{2}{3}\right)^4\left(\frac{1}{3}\right) + {}^5C_3\left(\frac{2}{3}\right)^3\left(\frac{1}{3}\right)^2 = 0.79,$$

$$P(RRB, 7) = {}^7C_7\left(\frac{2}{3}\right)^7 + {}^7C_6\left(\frac{2}{3}\right)^6\left(\frac{1}{3}\right) + {}^7C_5\left(\frac{2}{3}\right)^5\left(\frac{1}{3}\right)^2 + {}^7C_4\left(\frac{2}{3}\right)^4\left(\frac{1}{3}\right)^3 = 0.827,$$

and so on, showing how in the finite game the 2nd player holds a significant advantage.

In the short run, chance may seem to be volatile and unfair. Considering the misconceptions, inconsistencies, paradoxes and counter intuitive aspects of probability, it is not a surprise that as a civilization it has taken us a long time to develop some methods to deal with this. In antiquity, chance mechanisms, such as coins, dice and cards were used for decision making and there was a strong belief in the fact that God or Gods controlled the outcome. Even today, some people see chance outcomes as fate or destiny – "that which was meant to be".

Maurice Kendall points out that, man is in his childhood and is still afraid of the dark. Few prospects are darker than the future subject to blind chance! [12]❑

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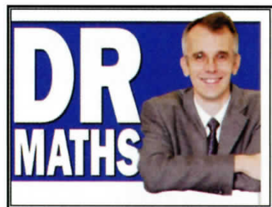

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