Why Are Kamei Diagrams Effective for Finding Divisors?

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Abstract

Divisors do not exist independently; they have a certain structure, which can be represented as a diagram in one, two, three, or n dimensions. This is called a "Kamei diagram," named after Kikuo Kamei, who discovered them in 1975. The twelve divisors of 60 are arranged at the vertices and grid points of a three-dimensional cuboid.

Keywords: Kamei diagrams, divisors, prime factor decomposition, hypercubes

1. Structure of divisor sets

A "divisor" is a number that divides a number leaving no remainder. For example, there are twelve divisors of 60, namely,

1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30, 60,

and because there are so many, it is easy to overlook one. We could divide 60 by each integer from 1 to 60 and take only those that divide evenly as its divisors, but performing sixty divisions is inefficient. Thus, we generally utilize prime factorization. Namely, we divide 60 starting with small prime numbers and express it as a product of primes, like so:

$$60 \div 2 = 30,$$

$$30 \div 2 = 15,$$

$$15 \div 3 = 5.$$

Thus, we can represent 60 as a product of the primes 2, 2, 3, and 5:

$$60 = 2 \times 2 \times 3 \times 5.$$

Any combination of these primes 2, 2, 3, and 5 will be a divisor. For example, there is one factor with no primes,

1

three factors with one prime,

2, 3, 5

four factors with two primes,

$$2 \times 2 = 4$$

$$2 \times 3 = 6$$

$$2 \times 5 = 10$$

$$3 \times 5 = 15$$

three factors with three primes,

$$2 \times 2 \times 3 = 12$$

$$2 \times 2 \times 5 = 20$$

$$2 \times 3 \times 5 = 30$$

and one factor with four primes,

$$2 \times 2 \times 3 \times 5 = 60.$$

Above, we found twelve combinations, so 60 has twelve divisors:

$$1 + 3 + 4 + 3 + 1 = 12.$$

However, this method, too, becomes increasingly cumbersome as the number of primes increases.

One method to ensure no divisors are missed is multiplying the numbers at either end to see if they equal 60:

$$1 \times 60 = 60$$
$$2 \times 30 = 60$$
$$3 \times 20 = 60$$
$$4 \times 15 = 60$$
$$5 \times 12 = 60$$
$$6 \times 10 = 60$$

$$1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 10 \quad 12 \quad 15 \quad 20 \quad 30 \quad 60$$

This method is not perfect, however, because it can only be applied after the divisors have been calculated.

But perhaps there's a relation between divisors? In fact, divisors do not exist independently; they have a certain structure, which can be represented as a diagram in one, two, three, or *n* dimensions. This is called a "Kamei diagram," named after Kikuo Kamei, who discovered them in 1975 [1]. Figure 1 shows an example, where the twelve divisors of 60 are arranged at the vertices and grid points of a three-dimensional cuboid.

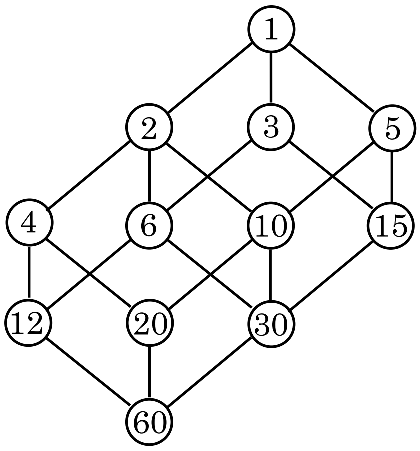


Figure 1. A Kamei diagram showing the divisors of 60.

2. Constructing Kamei diagrams

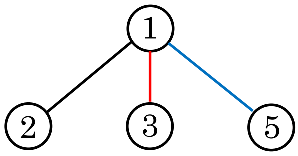
Let's see how this diagram is constructed by tracing through the prime factorization of 60:

$$60 = 2 \times 2 \times 3 \times 5.$$

The number 60 has four factors, but since the 2 is repeated, there are three unique prime factors. Thus, the Kamei diagram can be constructed as a three-dimensional graph.

Let's demonstrate that by connecting the divisors starting from 1, we reach 60. The directions corresponding to the mutually prime factors 2, 3, and 5 are associated with the three mutually perpendicular axes of a three-dimensional x, y, z coordinate system. We represent multiplying by 2 as moving along the x-axis, multiplying by 3 as moving along the y-axis, and multiplying by 5 as moving along the z-axis. These are respectively distinguished by black, red, and blue lines in the figure below, where we can progress from ① to ②, ③, or ⑤.

$$1 \times 2 = 2$$
$$1 \times 3 = 3$$
$$1 \times 5 = 5$$



From ②, the remaining factors are 2, 3, and 5, so three directions are possible from there:

$$2 \times 2 = 4$$
$$2 \times 3 = 6$$
$$2 \times 5 = 10$$

Thus, we proceed along black to ④, along red to ⑥, and along blue to ⑩.

From ③, the remaining factors are 2, 2, and 5, so two directions (2 and 5) are possible from there:

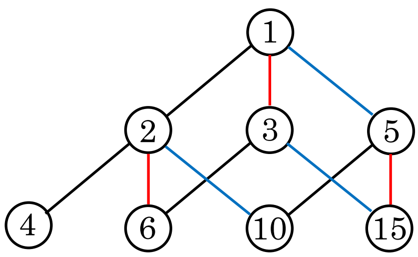
$$3 \times 2 = 6$$
$$3 \times 5 = 15$$

Hence, we proceed along black to ⑥, and along blue to ⑮.

From ⑤, the remaining factors are 2, 2, and 3, so two directions (2 and 3) are possible from there:

$$5 \times 2 = 10$$
$$5 \times 3 = 15$$

Therefore, we proceed along black to ⑩, and along red to ⑮.



Along the third row are ④, ⑥, ⑩, and ⑮. There are two directions we can advance in from ④:

$$4 \times 3 = 12$$
$$4 \times 5 = 20$$

So, we proceed along red to ⑫ and along blue to ⑳.

There are two directions we can proceed in from ⑥:

$$6 \times 2 = 12$$

$$6 \times 5 = 30$$

We thus proceed along black to ⑫ and along blue to ㉚.

There are two directions we can proceed in from ⑩:

$$10 \times 2 = 20$$

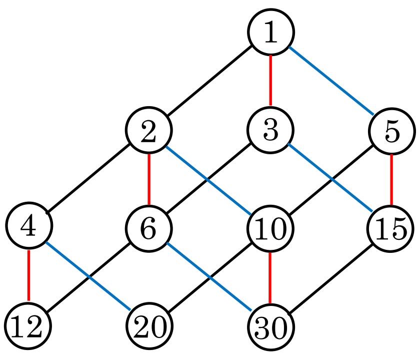
$$10 \times 3 = 30$$

Hence, we proceed along black to ⑳, and along red to ㉚.

There is only one direction from ⑮:

$$15 \times 2 = 30$$

We thus proceed along black to ㉚.

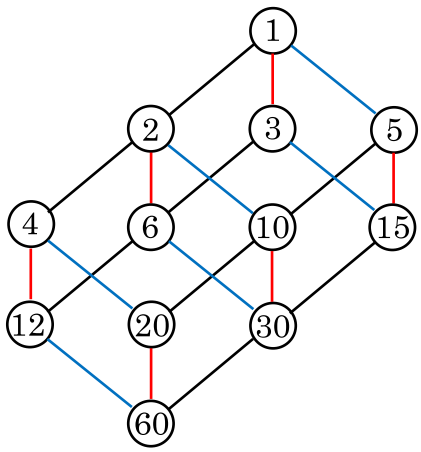


In this way, the fourth row has ⑫, ⑳, and ㉚. Each has only one direction in which we can proceed, all leading to 60:

$$12 \times 5 = 60$$

$$20 \times 3 = 60$$

$$30 \times 2 = 60$$



Having done this, we have constructed a Kamei diagram for the divisors of 60 shown in Fig. 1.

3. Higher-dimension Kamei diagrams

We can construct a Kamei diagram for the divisors of any natural number *N*. When the prime factorization of *N* produced three mutually prime factors, the result was a three-dimensional Kamei diagram. Similarly, four prime factors will result in a four-dimensional diagram, and five prime factors produce a five-dimensional diagram. For example, the divisors of 30 produce a three-dimensional diagram, while those of 210 and 2310 respectively produce four- and five-dimensional diagrams:

$$30 = 2 \times 3 \times 5$$
$$210 = 2 \times 3 \times 5 \times 7$$
$$2310 = 2 \times 3 \times 5 \times 7 \times 11$$

For four dimensions and above, Kamei diagrams can be drawn as hypercubes, and algorithms for doing so have been demonstrated [1].

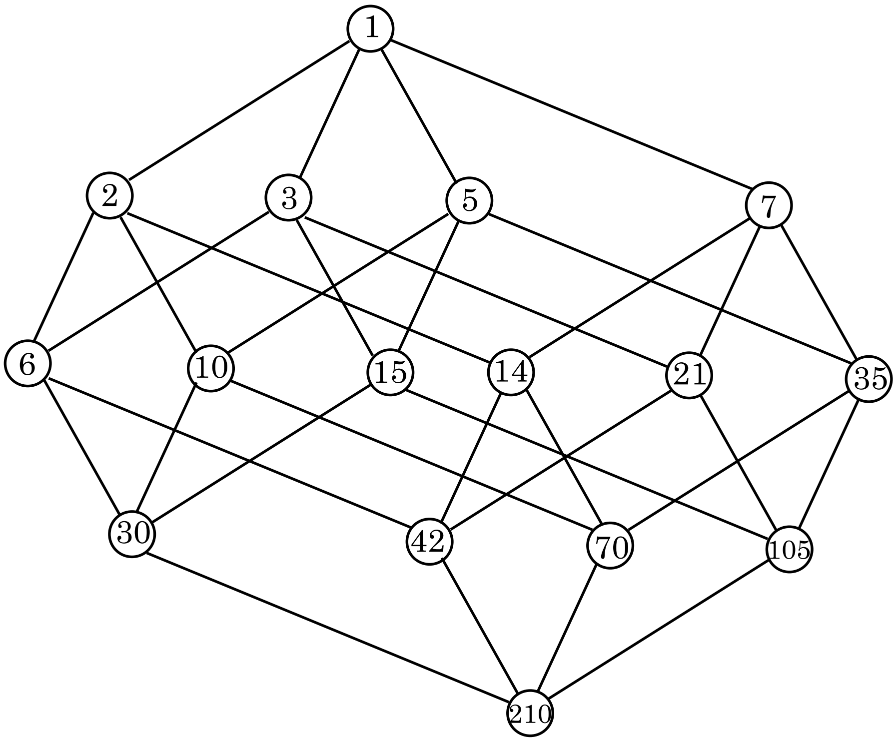


Figure 2. A four-dimensional Kamei diagram showing the divisors of 210.

4. Kamei diagrams and factors for numbers up to 100

Let's find the divisors for some numbers between 1 and 100 and draw their respective Kamei diagrams. Prime numbers have only two divisors: 1 and the prime itself. There are 25 primes up to 100, so excluding 1 and those 25 primes, there are 74 numbers for consideration. Among them, we will focus on eight numbers: 6, 16, 24, 30, 54, 60, 90, and 100 (Table 1).

As described above, we factorize the number *N* into its prime factors find its divisors by multiplying those factors together. When *N* is prime factorized as

$$N = p^i q^j r^k,$$

the number of divisors M is

$$M = (i+1)(j+1)(k+1).$$

A one-dimensional Kamei diagram can be drawn when there is one prime factor, a two-dimensional diagram for two mutually prime factors, and a three-dimensional diagram for three factors (Fig. 3).

For example, $N = 24$ is factorized as $2^3 \times 3$, and its divisors are 1, 2, 3, 4, 6, 8, 12, and 24. The number of divisors $M = (3+1) \times (1+1) = 8$. Since there are two mutually prime factors, the Kamei diagram will be two-dimensional.

References

[1] Kamei, Kikuo, "Laboratory Window: A Representation of Hypercubes," *Mathematical Sciences*, Jan 1992, pp. 68–73.

N	Prime decomposition	Divisors	Number of divisors	Kamei diagram dimensions
6	2×3	1, 2, 3, 6	4	2
16	2^4	1, 2, 4, 8, 16	5	1
24	$2^3 \times 3$	1, 2, 3, 4, 6, 8, 12, 24	8	2
30	$2 \times 3 \times 5$	1, 2, 3, 5, 6, 10, 15, 30	8	3
54	2×3^3	1, 2, 3, 6, 9, 18, 27, 54	8	2
60	$2^2 \times 3 \times 5$	1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30, 60	12	3
90	$2 \times 3^2 \times 5$	1, 2, 3, 5, 6, 9, 10, 15, 18, 30, 45, 90	12	3
100	$2^2 \times 5^2$	1, 2, 4, 5, 10, 20, 25, 50, 100	9	2

Table 1. Divisors for numbers up to 100 (excerpt).

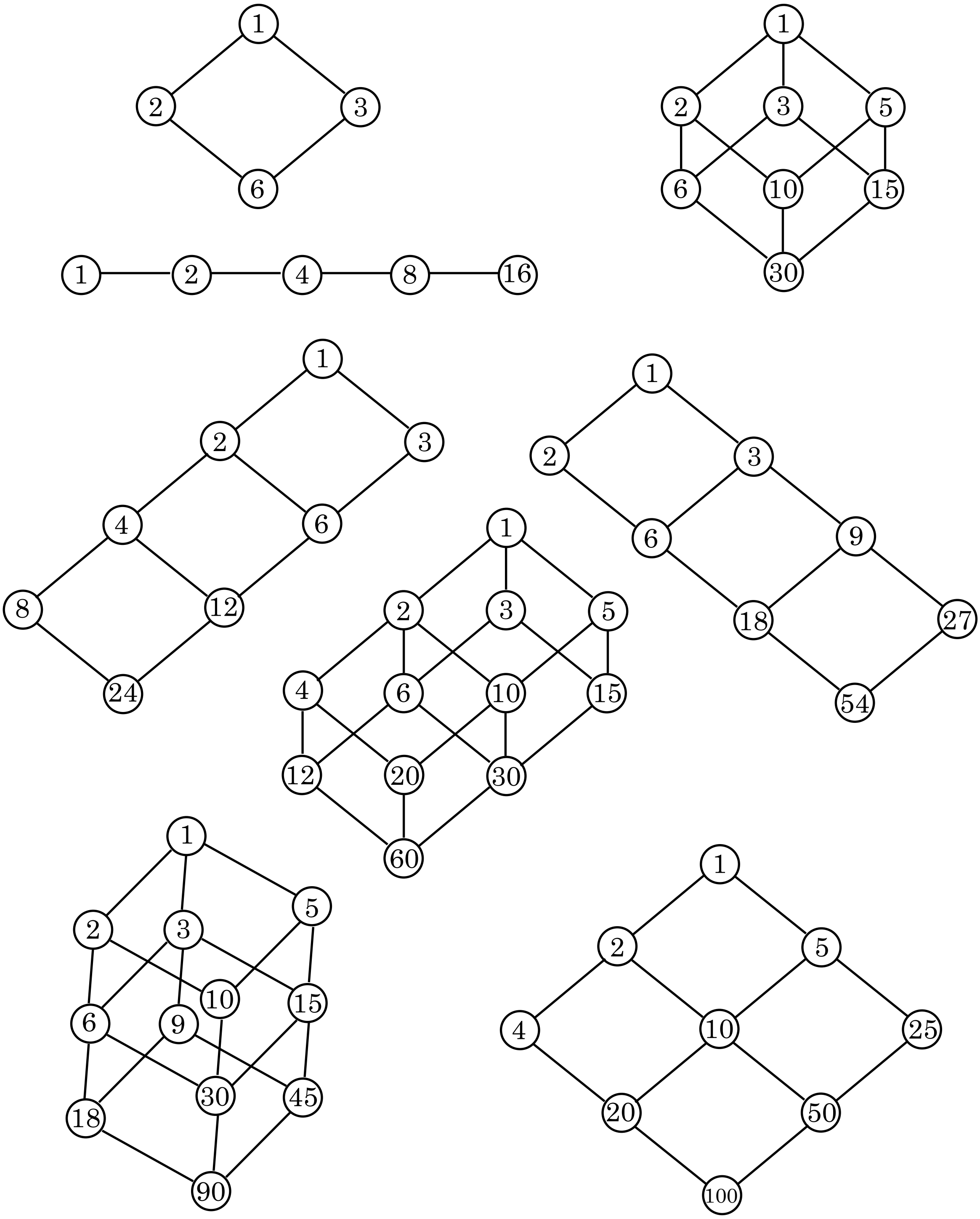


Fig. 3: Kamei diagrams for the divisors of 6, 16, 24, 30, 54, 60, 90, and 100.